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***In Memorian**

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APPLICATION OF CLIFFORD ALGEBRA IN SOLVING THE EIGEN EQUATIONS OF QUANTUM MECHANICS

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Abstract

Clifford algebra is a unified language and efficient tool for geometry and physics. In this paper, we derive the integrable conditions for Dirac and Pauli equations by Clifford algebra, which shows Clifford algebra has a standard operation procedure and deep insights into the structure of the equations. Here the integrable conditions are only regarded as algebraic conditions. The commutators expressed by Clifford algebra have a neat and covariant form, which are naturally valid in curvilinear coordinate system and curved space-time. For Pauli and Schrödinger equation, many solutions in axisymmetric potential and magnetic field are also integrable. By the virtue of Clifford algebra, the physical researches may be greatly promoted.

Keywords: Clifford algebra; Dirac equation; Pauli equation; Axisymmetric potential; Integrability.

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1 Introduction

The fermions are described by spinors and satisfy the Dirac equation or the non-relativistic approximation Schrödinger equation, and the energy eigen states in various potentials reflect the physical properties of the particles, so how to obtain exact eigensolutions of Dirac equation or Schrödinger equation with complicated potentials is an important issue subject. Numerous studies have been conducted on this subject. For the linear Schrödinger equation, Pauli equation and Dirac equation in spherical coordinate system (r, θ, φ) coupling with external potential $V(r)$ and $\vec{A}(r, \theta)$, there are a lot of rigorous eigen solutions obtained by separation of variables, e. g. [1]-[11]. Some rigorous solutions for the dark nonlinear spinor models were derived and analyzed in [12, 13, 14].

In the process to solve for eigenfunctions by separation of variables, we find that the following commutative relations play a key role[1, 14, 15],

$$[\hat{H}, \hat{J}_z] = [\hat{H}, \hat{K}] = [\hat{K}, \hat{J}_z] = 0, \tag{1}$$

where the commutators $\{\hat{J}_z(\partial_\varphi), \hat{K}(\partial_\theta, \partial_\varphi), \hat{H}\}$ are respectively angular momentum, spin-orbit coupling and total energy operators defined by

$$\hat{J}_z = -i\hbar\partial_\varphi + \frac{1}{2}\hbar S_3, \quad \hat{K} = \gamma^0(\hat{L}\cdot\vec{S} + \hbar), \quad \hat{H} = H(\partial_\varphi, \partial_\theta, \partial_r). \tag{2}$$

The commutators $\{\hat{J}_z, \hat{K}, \hat{H}\}$ form the complete chain of operators (1), which enable us to solve the eigen solutions by separation of variables, and then the original problem reduces to ordinary differential equations.

The logical foundation of the above method of separation variables is the following well-known theorem.

Theorem 1 *For two linear Hermitian operators A, B on a vector space with finite degeneracy, if they commute with each other $[A, B] = 0$, then they have common eigenvectors.*

For any Hamiltonian operator $\hat{H}$, if we can construct the following Hermitian operators set[14, 15],

$$\hat{H}_1 = H_1(\partial_1), \quad \hat{H}_2 = H_2(\partial_1, \partial_2), \quad \cdots, \quad \hat{H} = H_n(\partial_1, \partial_2, \cdots, \partial_n), \tag{3}$$

which form an Abelian Lie algebra

$$[\hat{H}_j, \hat{H}_k] = 0, \quad (j, k = 1, \cdots, n), \tag{4}$$

then (4) forms the integrable conditions of the eigen equation $\hat{H}\phi = E\phi$, and the eigen solutions can be completely solved by means of separating variables. So the existence of complete commutative operator chain forms the integrable condition for Dirac equation or Schrödinger equation.

Since the construction of commutative operator chain is quite simple and elementary, and the existence conditions are easily converted into some algebraic requirements for potentials as shown below, then the integrable condition of a quantum system is equivalently obtained. For example, in [10] a series of exact eigen solutions of Dirac equation were derived by introducing an additional invariant operator. The Dirac equation can be characterized by the set of independent commuting operators $\mathcal{H} = \{\hat{H}_1; \hat{H}_2; \cdots; \hat{H}_n\}$, where n corresponds to the number of the degrees of freedom, and where the set $\mathcal{H}$ includes the Hamiltonian and operators of the integrals of motion, i.e., invariants. In quantum case the operators of observables in such sets have a joint system of eigenstates, and their eigenvalues determine quantum numbers of each state. The operators of observables in such sets have a joint system of eigenstates, and their eigenvalues determine quantum numbers of each state. It is well known that in the relativistic Coulomb problem the Hamiltonian and the angular momentum do not generate a complete set of commuting observables, and an additional operator is needed to complete the set.

By (2) we find the commutative operators are usually related to some symmetries of the system which can be only displayed in a curvilinear coordinate system. For such problem, it will be very convenient to use Clifford algebra for discussion and calculation. Clifford algebra is a unified language for physics, which faithfully reflects the intrinsic symmetry and property of space-time and fields[14, 16, 17, 18]. We can exactly express dynamical equations in physics by Clifford algebra, and in the representation there is nothing superfluous or missed. The parameters in Clifford algebra all have clear physical meanings and the operations become simple and standard. In [19], Clifford algebra $C\ell_{3,0}$ is used to calculate the nuclear magnetic moments, and some precise and interesting results were obtained, which reflects that nucleons have complex internal structure. In contrast with the conventional and direct calculation in [15], the additional operator becomes a covariant and symmetrical one, therefore we can realize the superiority of Clifford algebra.

2 Eigen Equation Represented by Clifford Algebra

There are several definitions for Clifford algebra. Clifford algebra is also called geometric algebra. If the definition is directly related to geometric concepts, it will bring great convenience to the study and research of geometry[20, 21].

Definition 1 *Assume the element of an $n = p+q$ dimensional space-time $\mathbb{M}^{p,q}$ over $\mathbb{R}$ is given by*

$$d\mathbf{x} = \gamma_\mu dx^\mu = \gamma^\mu dx_\mu = \gamma_a \delta X^a = \gamma^a \delta X_a, \tag{5}$$

where γ_a is the local orthogonal frame and γ^a is the coframe. The space-time is endowed with distance $ds = |d\mathbf{x}|$ and oriented volumes dV_k calculated by

$$d\mathbf{x}^2 = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\delta X^a \delta X^b, \quad (6)$$

$$dV_k = d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_k = \gamma_{\mu\nu\cdots\omega}dx_1^\mu dx_2^\nu \cdots dx_k^\omega, \quad (7)$$

in which $1 \leq k \leq n$, the Minkowski metric in matrix form reads $(\eta_{ab}) = diag(I_p, -I_q)$, $g_{\mu\nu}$ is the Riemann metric of the space-time, and **Grassmann basis** *$\gamma_{\mu\nu\cdots\omega} = \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\omega \in \Lambda^k\mathbb{M}^{p,q}$. Then the following Clifford-Grassmann number with basis*

$$C = C_0 I + C_\mu\gamma^\mu + C_{\mu\nu}\gamma^{\mu\nu} + \cdots + C_{12\cdots n}\gamma^{12\cdots n}, \quad (\forall C_{\mathbf{k}} \in \mathbb{R}) \quad (8)$$

together with multiplication rule of basis given in (6) *and associativity define the 2^n-dimensional real universal* **Clifford algebra** *$C\ell_{p,q}$.*

Due to the skew-symmetry of exterior products, the summations in (8) such as $C_{\mu\nu}\gamma^{\mu\nu}$ often adopt the convention $\mu < \nu$. But sometimes we use other convenient forms, for example, in the 1+3 dimensional space-time we usually use the 3-dimensional vector forms (C_{01}, C_{02}, C_{03}) and (C_{23}, C_{31}, C_{12}). There is also complete anti-symmetry form $\frac{1}{2!}C_{\mu\nu}\gamma^{\mu\nu}, C_{\mu\nu} = -C_{\nu\mu}$, but the results are all the same.

For clearness, it is necessary to clarify the conventions and notations frequently used in the paper. We take $c = 1$ as unit of speed. The Einstein's summation is used. For index, we use the Greek characters for curvilinear coordinates, Latin characters for local Minkowski coordinates and $\{i, j, k, l\}$ for spatial indices. The Pauli matrices are expressed by

$$\sigma^a = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad (9)$$

$$\widetilde{\sigma}^a = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3), \quad \sigma^\mu = f^\mu_{\ a}\sigma^a, \quad \widetilde{\sigma}_\mu = f_\mu^{\ a}\widetilde{\sigma}_a, \quad (10)$$

where $a, \mu \in \{0,1,2,3\}$, f^{μ}_{a}, $f_{\mu}{}^{a}$ are the frame coefficients satisfying

$$f_{\mu}{}^{a} f_{\nu}{}^{b} \eta_{ab} = g_{\mu\nu},\ f^{\mu}_{a} f^{\nu}_{b} \eta^{ab} = g^{\mu\nu},\ \eta_{ab} = \eta^{ab} = \mathrm{diag}(1,-1,-1,-1). \quad (11)$$

For frame coefficients, the first index is always for curvilinear coordinate, and the second index for Minkowski index. In $1+3$ dimensional space-time, γ^a and γ^μ can be expressed by Dirac matrices

$$\gamma^a = \begin{pmatrix} 0 & \widetilde{\sigma}^a \\ \sigma^a & 0 \end{pmatrix},\ \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\ \vartheta^5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \quad (12)$$

In equivalent sense, γ^a forms the unique representation for generators of $C\ell_{1,3}$. In curved space-time $\gamma^\mu = f^{\mu}_{a}\gamma^a$ and $\gamma_\mu = f_{\mu}{}^{a}\gamma_a$. Since the Clifford algebra is isomorphic to matrix algebra[14, 20, 21], we need not distinguish matrix γ^a with tetrad γ^a. Thus we have

$$\begin{cases} \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}, & \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \\ \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}, & \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \end{cases} \quad (13)$$

In the case without confusion we omit the identity matrix I.

In (13), the product between basis such as $\gamma_a\gamma_b$ is called **Clifford product**, which is isomorphic to matrix product. In physics and geometry, The multiplication of variables are usually Clifford products, but only by projecting the variables onto Grassmann basis, their physical and geometrical meanings are clear. For basis of Clifford algebra, we have[14]

Theorem 2 *For the relations between Clifford products and Grassmann products, we have*

$$\gamma^\mu\gamma^{\theta_1\theta_2\cdots\theta_k} = \gamma^\mu \odot \gamma^{\theta_1\theta_2\cdots\theta_k} + \gamma^{\mu\theta_1\cdots\theta_k},$$
$$\gamma^{\theta_1\theta_2\cdots\theta_k}\gamma^\mu = \gamma^{\theta_1\theta_2\cdots\theta_k} \odot \gamma^\mu + \gamma^{\theta_1\cdots\theta_k\mu}.$$
$$\gamma^{\lambda_1\lambda_2\cdots\lambda_k}\gamma^{\theta_1\theta_2\cdots\theta_l} = \sum_{n=1}^{\min(k,l)} \gamma^{\lambda_1\lambda_2\cdots\lambda_k} \odot^n \gamma^{\theta_1\theta_2\cdots\theta_l} + \gamma^{\lambda_1\lambda_2\cdots\lambda_k\theta_1\theta_2\cdots\theta_l}.$$
$$\gamma_{a_1a_2\cdots a_{n-1}} = \epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_n},\ \gamma_{a_1a_2\cdots a_{n-k}} = \tfrac{1}{k!}\epsilon_{a_1a_2\cdots a_n}\gamma_{12\cdots n}\gamma^{a_{n-k+1}\cdots a_n}.$$

In which $\odot^k$ *is* **multi-inner product** *of Clifford algebra, which is defined as follows,*

$$\gamma^\mu \odot \gamma^\nu = \gamma^\mu \cdot \gamma^\nu = g^{\mu\nu}, \quad \gamma^{\mu\nu} \odot^2 \gamma^{\alpha\beta} = g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}.$$
$$\gamma^\mu \odot \gamma^{\theta_1\theta_2\cdots\theta_k} = g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} - g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + (-1)^{k+1}g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}.$$
$$\gamma^{\theta_1\theta_2\cdots\theta_k} \odot \gamma^\mu = (-1)^{k+1}g^{\mu\theta_1}\gamma^{\theta_2\cdots\theta_k} + (-1)^k g^{\mu\theta_2}\gamma^{\theta_1\theta_3\cdots\theta_k} + \cdots + g^{\mu\theta_k}\gamma^{\theta_1\cdots\theta_{k-1}}.$$
$$\gamma^{\mu\nu} \odot \gamma^{\alpha\beta} = g^{\mu\beta}\gamma^{\nu\alpha} - g^{\mu\alpha}\gamma^{\nu\beta} + g^{\nu\alpha}\gamma^{\mu\beta} - g^{\nu\beta}\gamma^{\mu\alpha}, \ \cdots.$$

In the curvilinear coordinate system, Dirac equation with electromagnetic potential and nonlinear potential is given by[14, 21, 22]

$$\gamma^\mu(\hat{p}_\mu + \frac{1}{2}\hbar\gamma^5\Omega_\mu)\phi + i\gamma^5 F_\vartheta\phi = (mc - F_\gamma)\phi, \tag{14}$$

in which the parameters and operators are defined as

$$\hat{p}_\mu = i\hbar(\partial_\mu + \Upsilon_\mu) - eA_\mu, \quad \Upsilon_\mu = \frac{1}{2}f^\nu{}_a(\partial_\mu f_\nu{}^a - \partial_\nu f_\mu{}^a),$$
$$\Omega^\alpha = -\frac{1}{2}(\epsilon^{dabc}f^\alpha{}_d f^\mu{}_a f^\nu{}_b)\partial_\mu f_\nu{}^e\eta_{ce} = \frac{1}{4\sqrt{g}}\epsilon^{\alpha\mu\nu\omega}\eta_{ab}f_\omega{}^a(\partial_\nu f_\mu{}^b - \partial_\mu f_\nu{}^b).$$

Υ_μ is the Keller connection and Ω_μ is Gu-Nester potential, which is a pseudo vector. If the metric can be diagonalized, we have $\Omega_\mu \equiv 0$. The nonlinear potential $F = F(\check{\gamma}, \check{\vartheta})$ is a function of quadratic scalar and pseudo scalar,

$$\check{\gamma} = \phi^+\gamma^0\phi, \quad \check{\vartheta} = \phi^+\vartheta^5\phi, \quad F_\gamma = \frac{\partial F}{\partial\check{\gamma}}, \quad F_\vartheta = \frac{\partial F}{\partial\check{\vartheta}}. \tag{15}$$

the coefficients in (14) belong to $\Lambda^0 \cup \Lambda^1 \cup \Lambda^3 \cup \Lambda^4 \subset C\ell_{1,3}$.

To get the Hamiltonian formalism of (14), the time t should be global cosmic time. That is, the Hamiltonian formalism can be clearly expressed only in Gu's natural coordinate system(NCS)[23]

$$ds^2 = g_{tt}dt^2 - \bar{g}_{kl}dx^k dx^l, \quad d\tau = \sqrt{g_{tt}}dt = f_t{}^0 dt, \quad dV = \sqrt{\bar{g}}d^3x. \tag{16}$$

In which ds is the 4-d length of line element, $d\tau$ is the Newton's absolute cosmic time element and dV is the absolute volume element of space at time

t. The NCS generally exists and the global simultaneity is unique. Only in NCS can we clearly establish the Hamiltonian formalism and calculate the Noether charges.

NCS is different from the Gaussian coordinate system that is valid only in the neighborhood of the initial Cauchy hypersurface. The isodistant translating hypersurface will deform soon, so that the metric $ds^2 = dt^2 + g_{kl}dx^k dx^l$ becomes invalid. NCS is also different from the Einstein's lift moving along a geodesic, namely the co-moving coordinate system, as this requires the lift to be an infinitesimal volume in curved space-time. While NCS holds unconditionally and globally, and its time is objective cosmic time. g_{tt} represents gravity and cannot be merged into the coordinate t.

In NCS, to lift and lower the index of a vector means

$$\Upsilon^0 = g^{00}\Upsilon_0, \quad \Upsilon^k = -g^{kl}\Upsilon_l, \quad g_{kl}g^{ln} = \delta^n_k. \tag{17}$$

Then the eigen equation of (14) can be rewritten in the following Hamiltonian formalism

$$i\alpha^0(\partial_t + \Upsilon_t)\phi = \hat{H}\phi, \tag{18}$$

$$\hat{H} = -\alpha^k\hat{p}_k + eA_0\alpha^0 + (mc - F_\gamma)\gamma^0 - F_\vartheta\vartheta^5 - \Omega_\mu\hat{S}^\mu, \tag{19}$$

where $\hat{H}$ is the Hamiltonian in curved space-time, and $(\alpha^\mu, \hat{S}^\mu)$ are respectively current and spin operators defined by

$$\alpha^\mu \equiv \gamma^0\gamma^\mu = \mathrm{diag}(\sigma^\mu, \widetilde{\sigma}^\mu), \quad \hat{S}^\mu \equiv \frac{1}{2}\hbar\alpha^\mu\gamma^5 = \frac{1}{2}\hbar\mathrm{diag}(\sigma^\mu, -\widetilde{\sigma}^\mu). \tag{20}$$

$\vec{S} \in \Lambda^3$ is the usual spin of Dirac bispinor.

To get the eigen state of energy of a particle, we must separate the drifting motion of the particle by a local Lorentz transformation to its comoving coordinate system. Then we get eigen state

$$\hat{H}\phi = i\alpha^0(\partial_t + \Upsilon_t)\phi = E\phi, \quad (E \in \mathbb{R}). \tag{21}$$

In this case of the spinor at energy eigen state, the Hamiltonian $\hat{H}$ in (21) has the following properties: 1° $\hat{H}$ is Hermitian, so its eigen values are all real. 2° $\hat{H}$ is a linear operator with respect to derivatives $\nabla_k = \partial_k + \Upsilon_k$. 3° Formally, in (21) $\hat{H} \in C\ell_{4,0}$ with the generators (γ^0, α^k). We introduce the generators of $C\ell_{4,0}$ as follows,

$$\vartheta^a = (\gamma^0, \alpha^k), \quad \vartheta_a = \delta_{ab}\vartheta^b, \quad \vartheta^a\vartheta^b + \vartheta^b\vartheta^a = 2\delta^{ab}, \tag{22}$$

$$\vartheta^\mu = t^\mu{}_a\vartheta^a, \quad \vartheta_\mu = t_\mu{}^a\vartheta_a, \quad \vartheta_\mu\vartheta_\nu + \vartheta_\nu\vartheta_\mu = 2g_{\mu\nu} = 2\mathrm{diag}(1, g_{kl}), \tag{23}$$

$$t_\mu{}^a t_\nu{}^b\delta_{ab} = g_{\mu\nu}, \quad t_\mu{}^a t^\mu{}_b = \delta^a_b, \quad t_\mu{}^a t^\nu{}_a = \delta^\nu_\mu. \tag{24}$$

The complete bases of $C\ell_{4,0}$ and their relations are given by

$$I, \quad \vartheta^a, \quad \vartheta^{ab} \equiv \vartheta^a \wedge \vartheta^b, \quad \vartheta^{abc} = -\epsilon^{abcd}\vartheta_d\vartheta^5, \quad \vartheta^{0123} = \vartheta^5. \tag{25}$$

By (22)-(25), the Hamiltonian can be rewritten in Clifford algebra $C\ell_{4,0}$ as

$$\hat{H} = eA_0\sqrt{g^{00}} + \vartheta^\mu\hat{P}_\mu + \frac{i\hbar}{2}\Omega_\mu\vartheta^{0\mu}\vartheta^5 + \frac{i\hbar}{2}\sqrt{g^{00}}\Omega_0\vartheta^0\vartheta^5 - F_\vartheta\vartheta^5, \tag{26}$$

in which the Clifford numbers

$$A_0 \in \Lambda^0, \quad \hat{P}_\mu \in \Lambda^1, \quad \vec{\Omega} \in \Lambda^2, \quad \Omega_0 \in \Lambda^3, \quad F_\vartheta \in \Lambda^4. \tag{27}$$

We find the grade of some quantities in hyperbolic system γ^μ have been changed in elliptic system ϑ^μ. For example, $(\vec{\Omega}, \vec{S}) \in \Lambda^3$ in system γ^μ have been converted into $(\vec{\Omega}, \vec{S}) \in \Lambda^2$ in system ϑ^μ. The 4-vector momentum $\hat{P}_\mu$ is redefined by

$$\hat{P}_\mu = (mc - F_\gamma, -i\hbar(\partial_k + \Upsilon_k) - eA_k). \tag{28}$$

In $\hat{P}_\mu$ the sign of eA_k should be changed, this is because the Minkowski metric $\eta^{kk} = -1$ in $C\ell_{1,3}$ has been converted into $\delta^{kk} = 1$ in $C\ell_{4,0}$. So we have $\hat{P}^k\phi = \hat{P}_k\phi \to mv_k\phi$.

In order to calculate the derivatives of Clifford numbers, we introduce the following Christoffel-like connections

$$\Theta^{\mu}_{\alpha\beta} = t^{\mu}{}_{a}\partial_{\alpha}t_{\beta}{}^{a}. \tag{29}$$

We have relations

$$\Gamma^{\alpha}_{\mu\alpha} = \Theta^{\alpha}_{\mu\alpha}, \quad \Upsilon_{\mu} = \frac{1}{2}(\Theta^{\alpha}_{\mu\alpha} - \Theta^{\alpha}_{\alpha\mu}). \tag{30}$$

For tetrad the derivatives are given by

$$\partial_{\alpha}\vartheta_{\beta} = \vartheta_{a}\partial_{\alpha}t_{\beta}{}^{a} = \vartheta_{\mu}t^{\mu}{}_{a}\partial_{\alpha}t_{\beta}{}^{a} = \vartheta_{\mu}\Theta^{\mu}_{\alpha\beta}, \tag{31}$$

$$\partial_{\alpha}\vartheta^{\mu} = \vartheta^{a}\partial_{\alpha}t^{\mu}{}_{a} = \vartheta^{\beta}t_{\beta}{}^{a}\partial_{\alpha}t^{\mu}{}_{a} = -\vartheta^{\beta}\Theta^{\mu}_{\alpha\beta}. \tag{32}$$

So we can define the **frame covariant derivatives** $\overline{\nabla}_{\alpha}$ for Clifford numbers as

$$\begin{aligned}
\partial_{\alpha}\mathbf{A} &= \partial_{\alpha}(A^{\mu}\vartheta_{\mu}) = \vartheta_{\mu}\overline{\nabla}_{\alpha}A^{\mu} = \vartheta_{\mu}\left(\partial_{\alpha}A^{\mu} + \Theta^{\mu}_{\alpha\beta}A^{\beta}\right) \\
&= \partial_{\alpha}(A_{\mu}\vartheta^{\mu}) = \vartheta^{\mu}\overline{\nabla}_{\alpha}A_{\mu} = \vartheta^{\mu}\left(\partial_{\alpha}A_{\mu} - \Theta^{\beta}_{\alpha\mu}A_{\beta}\right), \qquad (33) \\
\partial_{\alpha}\mathbf{N} &= \vartheta_{\mu\nu}\overline{\nabla}_{\alpha}N^{\mu\nu} = \vartheta_{\mu\nu}\left(\partial_{\alpha}N^{\mu\nu} + \Theta^{\mu}_{\alpha\beta}N^{\beta\nu} + \Theta^{\nu}_{\alpha\beta}N^{\mu\beta}\right) \\
&= \vartheta^{\mu\nu}\overline{\nabla}_{\alpha}N_{\mu\nu} = \vartheta^{\mu\nu}\left(\partial_{\alpha}N_{\mu\nu} - \Theta^{\beta}_{\alpha\mu}N_{\beta\nu} - \Theta^{\beta}_{\alpha\nu}N_{\mu\beta}\right), \qquad (34)
\end{aligned}$$

and so on.

3 Integrable Conditions for Dirac Equations

Now we take spherical coordinate system as example to show the concepts and the advantage of Clifford algebra. For static potential $A_0 = A_0(r,\theta)$ and magnetic field $\vec{B} = B_z\vartheta^z\vartheta^5$, we have $\vec{A} = A(r,\theta)\vartheta^{\varphi}$ and

$$(g_{\mu\nu}) = \mathrm{diag}(1,1,r^2,r^2\sin^2\theta), \quad (t_{\mu}{}^{a}) = \mathrm{diag}(1,1,r,r\sin\theta). \tag{35}$$

$$\Upsilon_{\mu} = (0,\tfrac{1}{r},\tfrac{1}{2}\cot\theta,0), \quad \Omega_{\mu} = 0. \tag{36}$$

$$\hat{P}_{\mu} = \left(mc - F_{\gamma}, -i\hbar(\partial_r + \tfrac{1}{r}), -i\hbar(\partial_{\theta} + \tfrac{1}{2}\cot\theta), -i\hbar\partial_{\varphi} - eA\right). \tag{37}$$

In the case of diagonal metric we have $\Omega^\mu = 0$, then the Hamiltonian becomes

$$\hat{H} = eA_0 + \vartheta^\mu \hat{P}_\mu - F_\vartheta \vartheta^5. \tag{38}$$

Then we simply have

$$\hat{L}_z = -i\hbar\partial_\varphi, \quad [\hat{H}, \hat{L}_z] = 0. \tag{39}$$

Different from (2), the spin $\hat{S}$ vanishes in the operator, this is because the spinor ϕ has been implicitly made the following spin-$\frac{1}{2}$ transformation[14],

$$\psi = \Pi\psi', \quad \widetilde{\psi} = \Pi^*\widetilde{\psi}', \tag{40}$$

in which

$$\Pi = \frac{1}{\sqrt{2}} \begin{pmatrix} i\exp\left[-\frac{i}{2}\left(\theta+\varphi+\frac{\pi}{2}\right)\right], & \exp\left[\frac{i}{2}\left(\theta-\varphi+\frac{\pi}{2}\right)\right] \\ -\exp\left[-\frac{i}{2}\left(\theta-\varphi+\frac{\pi}{2}\right)\right], & -i\exp\left[\frac{i}{2}\left(\theta+\varphi+\frac{\pi}{2}\right)\right] \end{pmatrix}$$

and $\Pi^* = \Pi$.

Now we look for the additional Hermitian commutator

$$\hat{T}(\partial_\theta, \partial_\varphi) = T^0 + T^\theta \hat{P}_\theta + T^\varphi \hat{P}_\varphi, \tag{41}$$

in which $(T^0, T^\theta, T^\varphi)$ are all Hermitian matrices. By $[\hat{L}_z, \hat{T}] = 0$, we get $\partial_\varphi \hat{T} = 0$. This means $(T^0, T^\theta, T^\varphi)$ should be independent of φ. We examine $[\hat{H}, \hat{T}] = 0$. For any eigen solution ϕ, we have $\hat{P}_\varphi \phi = (m_z\hbar - eA)\phi$. This means $\hat{P}_\varphi$ can be regarded as an ordinary function for any eigen solution. Then we have integrable condition as

$$\begin{aligned} 0 &= [\hat{H}, \hat{T}] = [\hat{H}, \hat{T}]_A - i\hbar\vartheta^\mu\partial_\mu\hat{T} + i\hbar T^\theta\partial_\theta\hat{H} \\ &= [\vartheta^r, T^\theta]\hat{P}_r\hat{P}_\theta + [\vartheta^\theta, T^\theta]\hat{P}_\theta^2 + [\vartheta^r, \hat{T}]\hat{P}_r + \cdots, \end{aligned} \tag{42}$$

where $[\hat{H},\hat{T}]_A$ stands for algebraic commutation taking ∂_μ as ordinary numbers. The notation $[\hat{H},\hat{T}]_A$ is convenient for calculation, especially for programming. Then we get integrable conditions

$$[\vartheta^r, T^\theta] = [\vartheta^\theta, T^\theta] = [\vartheta^r, \hat{T}] = \cdots = 0. \tag{43}$$

By Hermiticity of $\hat{T}$ and (25), T^θ can be represented by the following Clifford algebra

$$T^\theta = X + Y_\mu \vartheta^\mu + iZ_{\mu\nu}\vartheta^{\mu\nu} + iU_\mu \vartheta^\mu \vartheta^{0r\theta\varphi} + V\vartheta^{0r\theta\varphi}, \tag{44}$$

in which all coefficients are real functions of (r,θ) and $Z_{\mu\nu} = -Z_{\nu\mu}$. Then by Thm.2 and (35), we have

$$\begin{aligned}\frac{1}{2}[\vartheta^r, T^\theta] &= Y_\mu \vartheta^{r\mu} + iZ_{\mu\nu}\vartheta^r \odot \vartheta^{\mu\nu} + iU_\mu(\vartheta^r \odot \vartheta^\mu)\vartheta^{0r\theta\varphi} + V\vartheta^r \odot \vartheta^{0r\theta\varphi} \\ &= Y_\mu \vartheta^{r\mu} + 2iZ_{r\nu}g^{rr}\vartheta^\nu + iU_r g^{rr}\vartheta^{0r\theta\varphi} - Vg^{rr}\vartheta^{0\theta\varphi}. \end{aligned} \tag{45}$$

By $[\vartheta^r, T^\theta] = 0$ we get

$$\begin{aligned} &Y_0 = Y_\theta = Y_\varphi = Z_{0r} = Z_{r\theta} = Z_{r\varphi} = U_r = V = 0. \\ &T^\theta = X + Y_r\vartheta^r + iZ_{0\theta}\vartheta^{0\theta} + iZ_{0\varphi}\vartheta^{0\varphi} \\ &\qquad + iZ_{\theta\varphi}\vartheta^{\theta\varphi} + i(U_0\vartheta^0 + U_\theta\vartheta^\theta + U_\varphi\vartheta^\varphi)\vartheta^{0r\theta\varphi}. \end{aligned}$$

By $[\vartheta^\theta, T^\theta] = 0$ we get

$$\begin{aligned} &\frac{1}{2}[\vartheta^\theta, T^\theta] = Y_r\vartheta^{\theta r} + 2i(-Z_{0\theta}g^{\theta\theta}\vartheta^0 - Z_{0\varphi}\vartheta^{0\theta\varphi} + Z_{\theta\varphi}g^{\theta\theta}\vartheta^\varphi) + iU_\theta g^{\theta\theta}\vartheta^{0r\theta\varphi} = 0, \\ &T^\theta = X + i(U_0\vartheta^0 + U_\varphi\vartheta^\varphi)\vartheta^{0r\theta\varphi} = X + iU_0\vartheta^{r\theta\varphi} - iU_\varphi g^{\varphi\varphi}\vartheta^{0r\theta}. \end{aligned}$$

Similarly, by Clifford calculus of (42) we finally derive the integrable conditions as

$$\hat{T} = ir(\vartheta^{0r\theta}\hat{P}_\theta + \vartheta^{0r\varphi}\hat{P}_\varphi). \tag{46}$$

$$A_0 = V(r), \quad A = A(\theta), \quad \partial_\theta F_\gamma = 0, \quad F_\vartheta = 0. \tag{47}$$

According to

$$A = A(\theta), \quad r\vartheta^{0r\theta} = \vartheta^{012}, \quad r\vartheta^{0r\varphi} = \frac{1}{\sin\theta}\vartheta^{013}, \tag{48}$$

we find $\hat{T}$ is actually independent of r. $\hat{T} \in \Lambda^3$ is an additional invariant, whose physical meaning is still unclear. In Minkowski space-time the condition for vector potential becomes

$$\vec{A} = \vartheta^{\varphi} A_{\varphi} = \frac{1}{r} U(\theta)(-\sin\varphi, \cos\varphi, 0), \quad U \equiv \frac{A(\theta)}{\sin\theta}. \tag{49}$$

Since the bispinor ϕ always has a spin-axis, we certainly have a commutator ∂_{φ} for any eigen state for a single spinor. Now we consider the Dirac equation in coordinate system (t, r, z, φ) with metric

$$ds^2 = dt^2 - R^2 dr^2 - Z^2 dz^2 - \rho^2 d\varphi^2, \tag{50}$$

where (R, Z, ρ) are smooth functions of (r, z). We get the Keller connection as[21]

$$\sqrt{g} = RZ\rho, \quad \Upsilon_{\mu} = \frac{1}{2}\left(0, \frac{\partial_r(Z\rho)}{Z\rho}, \frac{\partial_z(R\rho)}{R\rho}, 0\right). \tag{51}$$

For vector potential $\vec{A} = A(r, z)\vartheta^{\varphi}$, we have

$$\hat{H} = eA_0 + \vartheta^{\mu}\hat{P}_{\mu} - F_{\vartheta}\vartheta^5, \tag{52}$$

$$\hat{P}_{\mu} = [mc - F_{\gamma}, -i\hbar(\partial_r + \Upsilon_r), -i\hbar(\partial_z + \Upsilon_z), -i\hbar\partial_{\varphi} - eA]. \tag{53}$$

Indeed, we have $[\hat{H}, \partial_{\varphi}] = 0$.

By calculation similar to (45)-(47), we find $F_{\theta} = 0$, the second commutator $\hat{T}$ also has the same form of (46),

$$\hat{T} = iK(\vartheta^{0rz}\hat{P}_z + \vartheta^{0r\varphi}\hat{P}_{\varphi}), \tag{54}$$

where $K = K(r, z)$ is a smooth real function to be determined. Substituting (52) and (54) into (42), we get the integrable conditions as follows:

1° $R = R(r), K = Z = Z(r)$. By a transformation $dr' = R(r)dr$, then in equivalent sense we have

$$R = 1, \quad K = Z = Z(r), \tag{55}$$

where $Z(r)$ is any given smooth function.

2° If $\rho = Z(r)f(z)$ and $f(z)$ is any given smooth function, we have conditions for other parameters,

$$\partial_z A_0 = \partial_z F_\gamma = \partial_r A = 0. \tag{56}$$

This is similar to (47) in spherical coordinate system.

3° In the general case of $\rho \neq Z(r)f(z)$, assuming $\partial_\varphi \phi = im_z\phi$, we have

$$\partial_z A_0 = \partial_z F_\gamma = 0, \quad A = \hbar m_z + \frac{\rho Q(z)}{Z}, \tag{57}$$

where $Q(z)$ is any given arbitrary functions.

Especially, in cylindrical coordinate system we have

$$R = K = Z = 1, \quad \rho = r, \quad \Upsilon_\mu = (0, \frac{1}{2r}, 0, 0), \tag{58}$$

and integrable condition becomes

$$A_0 = V(r), \quad eA = rQ(z) + \hbar m_z, \quad \partial_z F_\gamma = 0. \tag{59}$$

In Minkowski space-time, the condition for vector potential becomes

$$e\vec{A} = eA_\varphi \vartheta^\varphi = \left(Q(z) + \frac{\hbar m_z}{r}\right)(-\sin\varphi, \cos\varphi, 0), \tag{60}$$

which explicitly depends on m_z. So the integrable condition is state dependent. In this case, we have

$$\hat{T} = i\vartheta^{012}(-i\hbar\partial_z) - i\vartheta^{013}Q(z). \tag{61}$$

If $Q = \frac{1}{2}(2n-1)\hbar z^{-1}$, the eigen function of $\hat{T}\phi = \lambda_z \hbar \phi$ can be expressed by Bessel functions

$$\phi = [u(r) + v(r)\vartheta^5](a, a, b, -b)^T,$$
$$a = \sqrt{|z|}J_n(|\lambda_z|z), \quad b = \frac{i}{\lambda_z\sqrt{|z|}}\left[|\lambda_z z|J_{n+1}(|\lambda_z|z) - 2nJ_n(|\lambda_z|z)\right].$$

In which λ_z is determined by boundary condition and (u, v, E) by $\hat{H}\phi = E\phi$.

4 Integrable Conditions for Pauli Equation

From the calculation of the above section, we find that there are too many constraints for Dirac equation, one can hardly get exact solution for complicated potential. But for the Pauli and Schrödinger equations, the situation is better. We consider the non-relativistic approximation,

$$i\hbar\partial_t\phi = \hat{H}\phi, \quad \hat{H} = \frac{\hbar^2}{2m}(\sigma\cdot\hat{p})^2 + eV, \tag{62}$$

where σ_k forms the generators of Clifford algebra $C\ell_{3,0}$. Substituting $\hat{p} = -i\hbar\nabla - e\vec{A}$ into (62), omitting $\mathrm{O}(\vec{A}^2)$ term and using the Coulomb's gauge condition $\nabla\cdot\vec{A} = 0$, we get

$$\begin{aligned}\hat{H} &= -\frac{\hbar^2}{2m}\Delta + eV + \frac{ie\hbar}{2m}(\partial_k A_l - \partial_l A_k)\sigma^{kl} \\ &= -\frac{\hbar^2}{2m}\Delta + eV + \frac{ie\hbar}{2m}B^m(\epsilon_{klm}\sigma^{kl}) \\ &= -\frac{\hbar^2}{2m}\Delta + eV - \frac{e\hbar}{2m}B^k\sigma_k. \end{aligned}\tag{63}$$

By (62) and (63), we obtain the Pauli equation,

$$i\hbar\partial_t\phi = \hat{H}\phi, \quad \hat{H} = -\frac{\hbar^2}{2m}\Delta + eV - \frac{e\hbar}{2m}B^k\sigma_k. \tag{64}$$

In $C\ell_{3,0}$, the generators or frame σ_a ($a \in \{1,2,3\}$) satisfy the algebraic rules

$$\sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}, \quad \sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu = 2g_{\mu\nu}, \quad \sigma^a = \delta^{ab}\sigma_b.$$

We have the basis set

$$I \in \Lambda^0, \quad \sigma_a \in \Lambda^1, \quad \sigma_{ab} \equiv \sigma_a \wedge \sigma_b = i\epsilon_{abc}\sigma^c \in \Lambda^2, \quad \sigma_{abc} = i\epsilon_{abc} I \in \Lambda^3.$$

Now, we examine the axisymmetrical potential in the spherical coordinate system (r, θ, φ). In this case, we have

$$f_\mu{}^a = \mathrm{diag}(1, r, r\sin\theta), \quad \sqrt{g} = r^2 \sin\theta, \quad V = V(r,\theta), \quad \vec{A} = \vec{A}(r,\theta).$$

Using $B^\mu_{;\mu} = \nabla \cdot \vec{B} = 0$, we obtain

$$(\partial_r + \frac{2}{r})B^r + (\partial_\theta + \cot\theta)B^\theta = 0. \tag{65}$$

Since the magnetic field $\vec{B}$ in (64) is given, we need not express it in covariant form, otherwise the Pauli equation (64) becomes more complicated. Assuming $\vec{B}$ is axisymmetrical, then we have

$$\vec{B} = (B_\rho \cos\varphi, B_\rho \sin\varphi, B_z), \quad B^k\sigma_k = \begin{pmatrix} B_z & B_\rho e^{-i\varphi} \\ B_\rho e^{i\varphi} & -B_z \end{pmatrix},$$

in which B_ρ and B_z are functions of (r, θ). $B^k\sigma_k$ depends on φ, which is troublesome, hence we make a spin transformation

$$\phi = \begin{pmatrix} 0 & \exp(-\frac{i\varphi}{2}) \\ \exp(\frac{i\varphi}{2}) & 0 \end{pmatrix} \psi,$$

then we obtain the Pauli equation and Hamiltonian in the spherical coordinate system as

$$i\partial_t\psi = \hat{H}\psi, \ \hat{H} = \tfrac{\hbar^2}{2m}\left(-\Delta + \tfrac{1}{4r^2\sin^2\theta}\right) + eV - \tfrac{e\hbar}{2m}(B_\rho\sigma_1 + B_z\sigma_3), \tag{66}$$

$$\Delta = (\partial_r^2 + \tfrac{2}{r}\partial_r) - \tfrac{1}{r^2}\hat{L}^2, \quad \hat{L}^2 = -\left(\partial_\theta^2 + \cot\theta\partial_\theta + \tfrac{1}{\sin^2\theta}\partial_\varphi^2\right), \tag{67}$$

where $\hat{L}$ is the angular momentum operator and $\hat{L}_z = -i\partial_\varphi$.

Obviously, for an axisymmetric potential, we have

$$[\hat{H}, \hat{L}_z] = 0, \quad \partial_\varphi \leftrightarrow m_z i,$$

where m_z is an integer. We look for the following second-order Hermitian operator

$$\hat{T}(\partial_\theta, \partial_\varphi) = -(\hat{L}^2 + \frac{1}{4\sin^2\theta}) + T^0 - iT^\theta(\partial_\theta + \frac{1}{2}\cot\theta) - iT^\varphi\partial_\varphi.$$

By Hermitian of $\hat{T}$, we have

$$T^0 = X + Y_a\sigma^a, \quad T^\theta = P + Q_a\sigma^a, \quad T^\varphi = R + Z_a\sigma^a.$$

Using $[\hat{T}, i\partial_\varphi] = 0$, we find that all coefficients are independent of φ. Then we obtain the integrable condition as

$$\begin{aligned}[\hat{H}, \hat{T}] &= [\hat{H}, \hat{T}]_A - \frac{\hbar^2}{2m}\left(\Delta\hat{T} + 2(\partial_r\hat{T}\partial_r + \frac{1}{r^2}\partial_\theta\hat{T}\partial_\theta)\right) \\ &\quad + \left(\hat{L}^2\hat{H} - 2(\partial_\theta\hat{H})\partial_\theta\right) + iT^\theta\partial_\theta\hat{H} = 0, \end{aligned} \tag{68}$$

in which $[\hat{H}, \hat{T}]_A$ is also the commutative operation that takes differential operators as ordinary numbers. By straightforward calculation, we obtain $T^\theta = T^\varphi = Y_2 = 0$ and

$$\partial_r X = 0, \quad \partial_\theta X = -2me\hbar^{-2}r^2\partial_\theta V, \tag{69}$$

$$Y_1 B_z = Y_3 B_\rho, \ \partial_r Y_1 = \partial_r Y_3 = 0, \ \partial_\theta(Y_1, Y_3) = e\hbar^{-1}r^2\partial_\theta(B_\rho, B_z). \tag{70}$$

With the natural boundary condition, (69) can be easily solved, we have

$$V = \Phi(r) + \frac{1}{r^2}U(\theta), \qquad X = -\frac{2me}{\hbar^2}U(\theta).$$

The solution to (70) is given by

$$B_\rho = \frac{e}{\hbar r^2}Y_1(\theta), \quad B_z = kB_\rho, \quad Y_3 = kY_1(\theta), \tag{71}$$

where k is a constant. (65) or $\nabla \cdot \vec{B} = 0$ becomes

$$(r\sin\theta\partial_r + \cos\theta\partial_\theta + \frac{1}{\sin\theta})B_\rho + (r\cos\theta\partial_r - \sin\theta\partial_\theta)B_z = 0, \qquad (72)$$

Substituting (71) into (72), we obtain

$$Y_1 = \frac{C_1}{\sin\theta(k\sin\theta - \cos\theta)}, \quad B_\rho = \frac{eC_1}{\hbar r^2 \sin\theta(k\sin\theta - \cos\theta)}$$

The magnetic field is singular. The simplest case is $k \to \infty$,

$$B_\rho = 0, \quad B_z = \frac{eC_1}{\hbar r^2 \sin^2\theta},$$

or constant magnetic field

$$B_\rho = 0, \quad B_z = B, \quad A = \frac{1}{2}Br\sin\theta.$$

In this case, one component of ψ vanishes, and the Pauli equation (66) is reduced to Schrödinger equation.

5 Discussion and Conclusion

From the above discussion we learn that, physical variables expressed by Clifford algebra simultaneously display tetrad and parameters as shown in (60). Clifford algebra automatically arrange a large amount of information in order, and make parameters and relations with clear physical meanings. The representation in Clifford algebra has a neat form without superfluous or missed contents, which is also valid in curved space-time. The second commutator $\hat{T}$ of Dirac equation in (46) and (54) is simply a covariant operator. For Pauli and Schrödinger equations, Many solutions in axisymmetric potential $V(r,\theta)$ and magnetic field $\mathbf{B}(r,\theta)$ are also integrable. In this paper, the integrable condition is only regarded as an algebra condition, which is unnecessarily related to some complicated geometric symmetry.

By the insights of Clifford algebra, the physical researches may be greatly promoted.

In the spherical coordinate system, the above procedure is invertible, so equations (47) actually form the sufficient and necessary conditions of the separation of variables to Dirac equation with axisymmetry. The following cases were solved in previous works by many researchers: $\{V = V(r), A = F_\beta = 0, F_\gamma = G(r)\}$ are the cases solved in [2, 5, 6, 7, 8, 9]. $\{V = \frac{Ze}{r}, A = \frac{e}{r\sin\theta}(a - b\cos\theta), F_\beta = F_\gamma = 0\}$ is the case solved in [11]. $\{V = A = F_\beta = \partial_\theta F_\gamma = 0\}$ is the simplest nonlinear case discussed in [12, 14]. For the general case of (47), although the separation of variables is valid for the Dirac equation (18), but the solutions to the reduced ordinary differential equations usually cannot be expressed by elementary functions. So some auxiliary numerical computation is still necessary for complicated potentials.

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A STUDY OF A QUADRILATERAL INSCRIBED IN A CIRCLE

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Abstract

In this article we have studied some interesting properties of a Quadrilateral inscribed in a circle. We have shown using co-ordinate geometry that the point of intersection of the diagonals of such a quadrilateral and the point of intersection of any two opposite sides (when they are not parallel) of that quadrilateral when joined by a straight line, the straight line meets the circle at a point at which the line joining this point and the point of intersection of the other two opposite sides (when they too are not parallel) of the quadrilateral is tangent to the circle.

Keywords: Co-ordinate geometry, Projective geometry, Quadrilateral, Circle, Tangent

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1 Introduction:

The Theorem given in the next paragraph was solved by using co-ordinate geometry by the author in 1974. The author communicated this deduction to Shri Debabrata Ghosh (of the Reserve Bank of India, Kolkata), a distinguished mathematician and physicist of Kolkata at that time. Shri Ghosh wrote it to Prof. H.S.M. Coxeter (of the University of Toronto, Canada), the great geometer of all the time. In reply, Prof.Coxeter wrote to Shri Ghosh, "The Euler-Hazra theorem is quite well known, and easy to prove by the methods of projective geometry; see, for instance, Coxeter & Greitzer , Geometry Revisited (Random House, New York , 1967), pp. 76, 135, 136 and 145 (Theorem 6.51). In your notation, the point of contact of the two tangents from P lie on the polar of P, which is the line QE."[1]. I am attaching this letter for historical interest. However, I am giving my petty proof of the theorem in the following paragraphs as was sent to Prof. Coxeter by Shri Debabrata Ghosh.

The theorem has not been discussed in any relevant books in this subject as in [2-7] except in [8] where the theorem has not been stated as associated with Euler.

2 Theorem

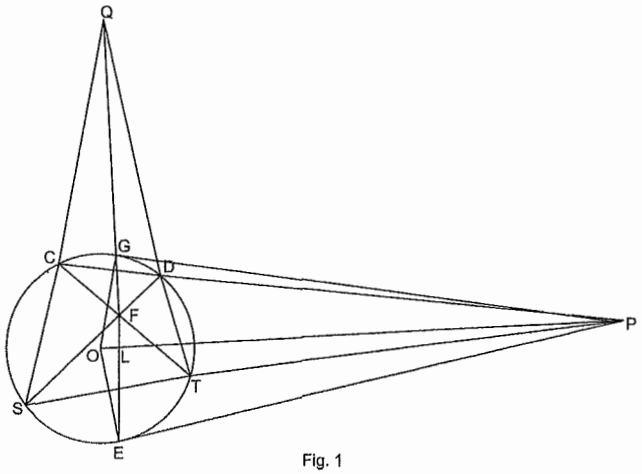

Fig. 1

The point of intersection of the diagonals of any quadrilateral inscribed in a circle and the point of intersection of any two opposite sides (when they are not parallel) of that quadrilateral when joined by a straight line, the straight line meets the circle at a point at which the line joining this point and the point of intersection of the other two opposite sides (when they too are not parallel) of the quadrilateral is tangent to the circle.

DCST is a quadrilateral inscribed in a circle of radius r. The diagonals TC and SD of the quadrilateral intersect at the point F. Two opposite sides of the quadrilateral DC and TS meet at P. The other opposite two sides of the quadrilateral meet at Q. Q F crosses the circle at G and at E. Prove that PG is a tangent to the circle at G and PE is a tangent to the circle at E .(Vide Fig.1)

3 Construction:

Let O be the center of the circle. Join OP. Let the straight line QGFE meet OP at L.

Let OP = a. Considering, the origin at O (0,0), and OP , the OX axis, let us construct OY axis. Suppose with reference of this plane co-ordinate axis P=(a ,0), S (x_1, y_1), T=(x_2 , y_2); C =(x_3 ,y_3); D=(x_4 , y_4); Q =(x_5 , y_5); F=(x_6 , y_6).

4 Commentary:

The necessary and sufficient conditions of proving the Theorem is to show that $x_5 = x_6 = r^2 / a$:

[G is an intermediate point on the straight line QF. If the x-coordinates of both the points Q and F are r^2/a, then x- coordinates of the point G will also

be r^2/a. The straight line QF meets OP, the x-axis at L. Therefore, the x-coordinate of the point L is r^2/a, i.e., the straight line QGFL is perpendicular to OP at L and the x-coordinate of the point E is too r^2/a.

OL= r^2/a, OP= a , OG =OE=r

$$PG^2 = GL^2 + PL^2 = r^2 - \frac{r^4}{a^2} + (a - \frac{r^2}{a})^2 = a^2 - r^2$$

Similarly,

$$PE^2 = EL^2 + PL^2 = r^2 - \frac{r^4}{a^2} + (a - \frac{r^2}{a})^2 = a^2 - r^2$$

Hence the above necessary and the sufficient condition]

It could be proved that

$$x_5 = \frac{x_3(x_2 - x_4)y_1 + x_4(x_3 - x_1)y_2 + x_1(x_4 - x_2)y_3 + x_2(x_1 - x_3)y_4}{(x_2 - x_4)y_1 + (x_3 - x_1)y_2 + (x_4 - x_2)y_3 + (x_1 - x_3)y_4}$$

(Vide calculation 1)

$$x_6 = \frac{x_4(x_2 - x_3)y_1 + x_3(x_4 - x_1)y_2 + x_2(x_1 - x_4)y_3 + x_1(x_3 - x_2)y_4}{(x_2 - x_3)y_1 + (x_4 - x_1)y_2 + (x_1 - x_4)y_3 + (x_3 - x_2)y_4}$$

(Vide calculation 2]

When expressed in terms of (x_1, y_1), (x_2, y_2); (x_3, y_3); (x_4, y_4) the necessary and sufficient condition of proving the proposition

$$\frac{x_3(x_2-x_4)y_1+x_4(x_3-x_1)y_2+x_1(x_4-x_2)y_3+x_2(x_1-x_3)y_4}{(x_2-x_4)y_1+(x_3-x_1)y_2+(x_4-x_2)y_3+(x_1-x_3)y_4}$$

$$=\frac{x_4(x_2-x_3)y_1+x_3(x_4-x_1)y_2+x_2(x_1-x_4)y_3+x_1(x_3-x_2)y_4}{(x_2-x_3)y_1+(x_4-x_1)y_2+(x_1-x_4)y_3+(x_3-x_2)y_4}=\frac{r^2}{a}$$

5 Proof:

Let the gradient of the straight line PTS=m_1 and the gradient of the straight line PDC=m_2. Hence (in the x-axis intercepted form) the equation of the straight line PTS, $y=m_1(x-a)$ and the equation of the straight line PDC, $y=m_2(x-a)$ and the equation of the circle =$x^2+y^2=r^2$.

Let us first express (x_1, y_1), (x_2 , y_2); (x_3 ,y_3); (x_4 , y_4) in terms of m_1 and m_2.

The point S and T are crossing points of the circle $x^2+y^2=r^2$ and the straight line in the x-axis intercepted form $y=m_1(x-a)$.

Hence

$$x_1=\frac{m_1^2a+\sqrt{m_1^2(r^2-a^2)+r^2}}{m_1^2+1} \qquad y_1=\frac{m_1}{m_1^2+1}(\sqrt{m_1^2(r^2-a^2)+r^5}-a)$$

$$x_2=\frac{m_1^2a-\sqrt{m_1^2(r^2-a^2)+r^2}}{m_1^2+1} \qquad y_2=-\frac{m_1}{m_1^2+1}(\sqrt{m_1^2(r^2-a^2)+r^5}+a)$$

The points C and D are the crossing points of the circle $x^2+y^2=r^2$ and the straight line

$$y=m_2(x-a)$$

Hence

$$x_3 = \frac{m_2^2 a + \sqrt{m_2^2(r^2 - a^2) + r^2}}{m_2^2 + 1} \qquad y_{.3} = \frac{m_2}{m_2^2 + 1}(\sqrt{m_2^2(r^2 - a^2) + r^5} - a)$$

$$x_4 = \frac{m_2^2 a - \sqrt{m_2^2(r^2 - a^2) + r^2}}{m_2^2 + 1} \qquad y_{.4} = -\frac{m_2}{m_2^2 + 1}(\sqrt{m_2^2(r^2 - a^2) + r^5} + a)$$

Let us now abbreviate the above expressions in terms of l_1, l_2, A and B such that

$$l_1 = m_1^2(r^2 - a^2) + r^2 \qquad l_2 = m_2^2(r^2 - a^2) + r^2 \qquad A = \frac{1}{1 + m_1^2}$$

$$B = \frac{1}{1 + m_2^2}$$

Table of Abbreviation

$$x_1 = A(am_1^2 + \sqrt{l_1}) \qquad x_2 = A(am_1^2 - \sqrt{l_1})$$

$$x_3 = B(am_2^2 + \sqrt{l_2}) \qquad x_4 = B(am_2^2 - \sqrt{l_2})$$

$$y_1 = Am_1(\sqrt{l_1} - a) \qquad y_2 = -Am_1(\sqrt{l_1} + a)$$

$$y_3 = Bm_2(\sqrt{l_2} - a) \qquad y_4 = -Bm_2(\sqrt{l_2} + a)$$

Now Put

$$k = Aam_1^2 - Bam_2^2$$

and transform the denominators and the numerators as per the above abbreviations.

Now start with the following first equation

$$\frac{x_3(x_2-x_4)y_1+x_4(x_3-x_1)y_2+x_1(x_4-x_2)y_3+x_2(x_1-x_3)y_4}{(x_2-x_4)y_1+(x_3-x_1)y_2+(x_4-x_2)y_3+(x_1-x_3)y_4}$$

Denominator (Vide Calculation 3)

$$2(kAm_1+A^2am_1-ABam_2)\sqrt{l_1}\ -2(kBm_2-B^2am_2-ABam_1)\sqrt{l_2}$$

$$=2(\alpha\sqrt{l_1}-\beta)\sqrt{l_2}$$

$$[\alpha=kAm_1+A^2am_1-ABam_2=a\frac{(m_1-m_2)}{(1+m_1^2)(1+m_2^2)}\text{(Vide Calculation 7a)}$$

$$\beta=kBm_2-B^2am_2+ABam_1=a\frac{(m_1-m_2)}{(1+m_1^2)(1+m_2^2)}\text{(Vide Calculation 7b)]}$$

$$=2a\frac{(m_1-m_2)}{(1+m_1^2)(1+m_2^2)}(\sqrt{l_1}-\sqrt{l_2})$$

Numerator (Vide Calculation 4)

$$2AB(Aa^2m_1m_2^2+kam_1m_2^2+Bl_2m_1-Aa^2m_1^2m_2+kam_2-Bl_2m_2)\sqrt{l_1}$$

$$-2AB(Ba^2m_1m_2^2+kam_1^2m_2+Al_1m_1-Ba^2m_1^2m_2+kam_1-Al_1m_2)\sqrt{l_2}$$

$$=2AB(\gamma\sqrt{l_1}-\delta\sqrt{l_2})$$

$$[\gamma=(Aa^2m_1m_2^2+kam_1m_2^2+Bl_2m_1-Aa^2m_1^2m_2+kam_2-Bl_2m_2)$$

$$=(m_1-m_2)r^2\text{(Vide Calculation 8a)}$$

$$\delta = (Ba^2 m_1 m_2^2 + kam_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + kam_1 - Al_1 m_2)]$$

$$= (m_1 - m_2) r^2 \text{ (Vide Calculation 8b)}$$

$$= 2r^2 \frac{(m_1 - m_2)}{(1+m_1^2)(1+m_2^2)} (\sqrt{l_1} - \sqrt{l_2})$$

Numerator/ denominator

$$= \frac{r^2}{a}$$

The second equation

$$\frac{x_4(x_2 - x_3)y_1 + x_3(x_4 - x_1)y_2 + x_2(x_1 - x_4)y_3 + x_1(x_3 - x_2)y_4}{(x_2 - x_3)y_1 + (x_4 - x_1)y_2 + (x_1 - x_4)y_3 + (x_3 - x_2)y_4}$$

(Vide Calculation 2)

Denominator (Vide Calculations 5)

$$2(kam_1 + A^2 am_1 - ABam_2)\sqrt{l_1} + 2(kBm_2 - B^2 am_2 + ABam_1)\sqrt{l_2}$$

$$= 2(\alpha\sqrt{l_1} + \beta\sqrt{l_2}) = 2a \frac{(m_1 - m_2)}{(1+m_1^2)(1+m_2^2)} (\sqrt{l_1} + \sqrt{l_2})$$

Numerator (Vide Calculation 6)

$$2AB(Aa^2 m_1 m_2^2 + kam_1 m_2^2 + Bl_2 m_1 - Aa^2 m_1^2 m_2 + kam_2 - Bl_2 m_2)\sqrt{l_1}$$

$$+2AB(Ba^2 m_1 m_2^2 + kam_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + kam_1 - Al_1 m_2)\sqrt{l_2}$$

$$= 2AB(\gamma\sqrt{l_1} + \delta\sqrt{l_2})$$

$$= 2r^2 \frac{(m_1 - m_2)}{(1+m_1^2)(1+m_1^2)}(\sqrt{l_1} + \sqrt{l_2})$$

Numerator/ denominator

$$= \frac{r^2}{a}$$ (Vide calculations under 7 & 8)

Hence the Theorem (QED).

Calculation 1

Q is the meeting point of the straight lines DT and SC.

The equation of the straight line DT (in the x-axis intercepted form)

$$y = \frac{y_2 - y_4}{x_2 - x_4}x - \frac{y_2 - y_4}{x_2 - x_4}x_2 + y_2$$

The equation of the straight line SC (in the x-axis intercepted form)

$$y = \frac{y_1 - y_3}{x_1 - x_3}x - \frac{y_1 - y_3}{x_1 - x_3}x_1 + y_1$$

Therefore

$$(\frac{y_2 - y_4}{x_2 - x_4} - \frac{y_1 - y_3}{x_1 - x_3})x + \frac{y_1 - y_3}{x_1 - x_3}x_1 - \frac{y_2 - y_4}{x_2 - x_4}x_2 + y_2 - y_1 = 0$$

$$x = \frac{x_3(x_2 - x_4)y_1 + x_4(x_3 - x_1)y_2 + x_1(x_4 - x_2)y_3 + x_2(x_1 - x_3)y_4}{(x_2 - x_4)y_1 + (x_3 - x_1)y_2 + (x_4 - x_2)y_3 + (x_1 - x_3)y_4}$$

Calculation 2

F is the crossing point of the of the straight line SD and TC

The equation of the straight line SD (in the x-axis intercepted form)

$$y = \frac{y_1 - y_4}{x_1 - x_4} x - \frac{y_1 - y_4}{x_1 - x_4} x_1 + y_1$$

The equation of the straight line TC (in the x-axis intercepted form)

$$y = \frac{y_2 - y_3}{x_2 - x_3} x - \frac{y_2 - y_3}{x_2 - x_3} x_2 + y_2$$

$$x = \frac{x_4(x_2 - x_3)y_1 + x_3(x_4 - x_1)y_2 + x_2(x_1 - x_4)y_3 + x_1(x_3 - x_2)y_4}{(x_2 - x_3)y_1 + (x_4 - x_1)y_2 + (x_1 - x_4)y_3 + (x_3 - x_2)y_4}$$

Calculation 3

$$(x_2 - x_4) = k - A\sqrt{l_1} + B\sqrt{l_2}$$

$$\times y_1 = Am_1(\sqrt{l_1} - a)$$

(Multiplication)

$$(x_2 - x_4)y_1 = Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2} + B\sqrt{l_1 l_2} - Al_1 - ak] \quad \text{(i)}$$

Similarly,

$$(x_3 - x_1) = -k - A\sqrt{l_1} + B\sqrt{l_2}$$

$$\times y_2 = -Am_1(\sqrt{l_1} + a)$$

$$(x_3 - x_1)y_2 = Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2} - B\sqrt{l_1 l_2} + Al_1 + ak] \quad \text{(ii)}$$

(i)+(ii)

$$= 2Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2}]$$

$$(x_4 - x_2) = -k + A\sqrt{l_1} - B\sqrt{l_2}$$

$$\times y_3 = Bm_2(\sqrt{l_2} - a)$$

$$(x_4 - x_2)y_3 = Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2} + A\sqrt{l_1 l_2} - Bl_2 + ak] \quad \text{(iii)}$$

$$(x_1 - x_3) = k + A\sqrt{l_1} - B\sqrt{l_2}$$

$$\times y_4 = -Bm_2(\sqrt{l_2} + a)$$

$$(x_1 - x_3)y_4 = Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2} - A\sqrt{l_1 l_2} + Bl_2 - ak] \quad \text{(iv)}$$

(iii)+(iv)

$$= 2Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2}]$$

Adding 3 (i)+3(ii)+3(iii)+3(iv) we have

$$(x_2 - x_4)y_1 + (x_3 - x_1)y_2 + (x_4 - x_2)y_3 + (x_1 - x_3)y_4$$

$$= 2Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2}] + 2Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2}]$$

$$= 2(kAm_1 + A^2am_1 - ABam_2)\sqrt{l_1} - 2(kBm_2 - B^2am_2 + ABam_1)\sqrt{l_2}$$

$$= 2(\alpha\sqrt{l_1} - \beta\sqrt{l_2})$$

$$[\alpha = kAm_1 + A^2am_1 - ABam_2 \qquad \beta = kBm_2 - B^2am_2 + ABam_1]$$

Calculation 4

$$(x_2 - x_4)y_1 = Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2} + B\sqrt{l_1l_2} - Al_1 - ak]$$

$$\times x_3 = B(am_2^2 + \sqrt{l_2})$$

$$x_3(x_2 - x_4)y_1 = ABm_1[(kam_2^2 + Aa^2m_2^2 + Bl_2)\sqrt{l_1}$$

$$-(Ba_2^2m_2^2 + Al_1 + ak)\sqrt{l_2} + (Bam_2^2 + k + Aa)\sqrt{l_1l_2} - Aal_1m_2^2 - a^2km_2^2 + Bal_2] \quad \text{(i)}$$

$$(x_3 - x_1)y_2 = Am_1[(k + Aa)\sqrt{l_1} - Ba\sqrt{l_2} - B\sqrt{l_1l_2} + Al_1 + ak]$$

$$\times x_4 = B(am_2^2 - \sqrt{l_2})$$

$$x_4(x_3 - x_1)y_4 = ABm_1[(Aa^2m_2^2 + kam_2^2 + Bl_2)\sqrt{l_1}$$

$$-(Ba^2m_2^2 + Al_1 + ak)\sqrt{l_2} - (Bam_2^2 + k + Aa)\sqrt{l_1l_2} + Aal_1m_2^2 + a^2km_2^2 + Bal_2] \text{(ii)}$$

(i)+(ii)

$$= 2ABm_1[(am_2^2k + Aa^2m_2^2 + Bl_2)\sqrt{l_1} - (a^2Bm_2^2 + ak + Al_1)\sqrt{l_2}]$$

$$(x_4 - x_2)y_3 = Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2} + A\sqrt{l_1l_2} - Bl_2 + ak]$$

$$\times x_1 = A(am_1^2 + \sqrt{l_1})$$

$$x_1(x_4 - x_2)y_3 = ABm_2[-Aa^2m_1^2 + ak - Bl_2)\sqrt{l_1}$$

$$+(Ba^2m_1^2 - kam_1^2 + al_1)\sqrt{l_2} + (Aam_1^2 - k + Ba)\sqrt{l_1l_2} + ka^2m_1^2 - Bam_1^2l_2 - Aal_1] \text{(iii)}$$

$$(x_1 - x_3)y_4 = Bm_2[-Aa\sqrt{l_1} + (Ba - k)\sqrt{l_2} - A\sqrt{l_1l_2} + Bl_2 - ak]$$

$$\times x_2 = A(am_1^2 - \sqrt{l_1})$$

$$x_2(x_1 - x_3)y_4 = ABm_2[-Aa^2m_1^2 + ak - Bl_2)\sqrt{l_1}$$

$$+(Ba^2m_1^2 - kam_1^2 + Al_1)\sqrt{l_2} - (Aam_1^2 - k + Ba)\sqrt{l_1l_2} - ka^2m_1^2 + Bam_1^2l_2 + Aal_1]$$

(iv)

(iii)+ (iv)

$$2ABm_2[(-Aa^2m_1^2 + ak - Bl_2)\sqrt{l_1} - (akm_1^2 - a^2Bm_1^2 - Al_1)\sqrt{l_2}]$$

Adding 4(i), 4(ii), 4(iii) and 4 (iv) we have,

$$x_3(x_2 - x_4)y_1 + x_4(x_3 - x_1)y_4 + x_1(x_4 - x_2)y_3 + x_2(x_1 - x_3)y_4$$

$$= 2ABm_1[(am_2^2k + Aa^2m_2^2 + Bl_2)\sqrt{l_1} - (a^2Bm_2^2 + ak + Al_1)\sqrt{l_2}]$$

$$+2ABm_2[(-Aa^2m_1^2 + ak - Bl_2)\sqrt{l_1} - (akm_1^2 - a^2Bm_1^2 - Al_1)\sqrt{l_2}]$$

$$= 2AB[(Aa^2m_1m_2^2 + kam_1m_2^2 + Bm_1l_2 - Aa^2m_1^2m_2 + kam_2 - Bl_2m_2)\sqrt{l_1}$$

$$-(Ba^2m_1m_2^2 + kam_1^2m_2 + Al_1m_1 - Ba^2m_1^2m_2 + kam_1 - Al_1m_2)\sqrt{l_2}] \text{ (v)}$$

$$= 2AB(\gamma\sqrt{l_1} - \delta\sqrt{l_2})$$

$$[\gamma = (Aa^2m_1m_2^2 + kam_1m_2^2 + Bl_2m_1 - Aa^2m_1^2m_2 + kam_2 - Bl_2m_2)$$

$$\delta = (Ba^2m_1m_2^2 + kam_1^2m_2 + Al_1m_1 - Ba^2m_1^2m_2 + kam_1 - Al_1m_2)]$$

Calculation 5

$$(x_2 - x_3) = k - A\sqrt{l_1} - B\sqrt{l_2}$$

$$\times y_1 = Am_1(\sqrt{l_1} - a)$$

$$(x_2 - x_3)y_1 = Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2} - B\sqrt{l_1l_2} - Al_1 - ak] \text{ (i)}$$

$$(x_4 - x_1) = -k - A\sqrt{l_1} - B\sqrt{l_2}$$

$$\times y_2 = -Am_1(\sqrt{l_1} + a)$$

$$(x_4 - x_1)y_2 = Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2} + B\sqrt{l_1l_2} + Al_1 + ak] \text{ (ii)}$$

Adding (i) and (ii)

$$= 2Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2}]$$

$$(x_1 - x_4) = k + A\sqrt{l_1} + B\sqrt{l_2}$$

$$\times y_3 = Bm_2(\sqrt{l_2} - a)$$

$$(x_1 - x_4)y_3 = Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2} + A\sqrt{l_1l_2} + Bl_2 - ak] \text{ (iii)}$$

$$(x_3 - x_2) = k - A\sqrt{l_1} - B\sqrt{l_2}$$

$$\times y_4 = -Bm_2(\sqrt{l_2} + a)$$

$$(x_3 - x_2)y_4 = Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2} - A\sqrt{l_1 l_2} - Bl_2 + ak] \text{ (iv)}$$

Adding (iii) and (iv)

$$= 2Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2}]$$

Adding 5(i), 5(ii) , 5(iii) and 5 (iv) we have,

$$(x_2 - x_3)y_1 + (x_4 - x_1)y_2 + (x_1 - x_4)y + (x_3 - x_2)y_4$$

$$= 2Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2}] + 2Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2}]$$

$$= 2[(kAm_1 + A^2am_1 - ABam_2)\sqrt{l_1} + (kBm_2 - B^2Am_2 + ABam_1)\sqrt{l_2}] \text{ (v)}$$

$$= 2\alpha\sqrt{l_1} + 2\beta\sqrt{l_2}$$

$$= 2(\alpha\sqrt{l_1} + \beta\sqrt{l_2})$$

Calculation 6

$$(x_2 - x_3)y_1 = Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2} - B\sqrt{l_1 l_2} - Al_1 - ak]$$

$$\times x_4 = B(am_2^2 - \sqrt{l_2})$$

$$ABm_1[(Aa^2m_2^2 + kam_2^2 + Bl_2)\sqrt{l_1} + (Ba^2m_2^2 + ka + Al_1)\sqrt{l_2}$$

$$-(Bam_2^2 + k + Aa)\sqrt{l_1 l_2} - Aal_1 m_2^2 - ka^2 m_2^2 - Bal_2] \text{ (i)}$$

$$(x_4 - x_1)y_2 = Am_1[(k + Aa)\sqrt{l_1} + Ba\sqrt{l_2} + B\sqrt{l_1 l_2} + Al_1 + ak]$$

$$\times x_3 = B(am_2^2 + \sqrt{l_2})$$

$$ABm_1[(Aa^2 m_2^2 + kam_2^2 + Bl_2)\sqrt{l_1} + (Ba^2 m_2^2 + ka + Al_1)\sqrt{l_2}$$

$$+(Bam_2^2 + k + Aa)\sqrt{l_1 l_2} + Aal_1 m_2^2 + ka^2 m_2^2 + Bal_2] \text{ (ii)}$$

Adding (i) and (ii)

$$= 2ABm_1[(Aa^2 m_2^2 + kam_2^2 + Bl_2)\sqrt{l_1} + (Ba^2 m_2^2 + ka + Al_1)\sqrt{l_2}$$

$$(x_1 - x_4)y_3 = Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2} + A\sqrt{l_1 l_2} + Bl_2 - ak]$$

$$\times x_2 = A(am_1^2 + \sqrt{l_1})$$

$$ABm_2[(-Aa^2 m_1^2 + ka - Bl_2)\sqrt{l_1} + (-Ba^2 m_1^2 + kam_1^2 - Al_1)\sqrt{l_2}$$

$$+(Aam_1^2 - k + Ba)\sqrt{l_1 l_2} + Bam_1^2 l_2 - ka^2 m_1^2 + Aal_1] \text{ (iii)}$$

$$(x_3 - x_2)y_4 = Bm_2[-Aa\sqrt{l_1} + (k - Ba\sqrt{l_2} - A\sqrt{l_1 l_2} - Bl_2 + ak]$$

$$\times x_1 = A(am_1^2 + \sqrt{l_1})$$

$$ABm_2[(-Aa^2 m_1^2 + ka - Bl_2)\sqrt{l_1} + (-Ba^2 m_1^2 + kam_1^2 - Al_1)\sqrt{l_2}$$

$$-(Aam_1^2 - k + Ba)\sqrt{l_1 l_2} - Bam_1^2 l_2 + ka^2 m_1^2 - Aal_1] \text{ (iv)}$$

Adding (iii) and (iv)

$$= 2ABm_2[(-Aa^2m_1^2 + ka - Bl_2)\sqrt{l_1} + (-Ba^2m_1^2 + kam_1^2 - Al_1)\sqrt{l_2}$$

Adding (i), (ii) , (iii) and (iv)

$$x_4(x_2 - x_3)y_1 + x_3(x_4 - x_1)y_2 + x_2(x_1 - x_4)y_3 + x_1(x_3 - x_2)y_4$$

$$= 2ABm_1[(Aa^2m_2^2 + kam_2^2 + Bl_2)\sqrt{l_1} + (Ba^2m_2^2 + ka + Al_1)\sqrt{l_2}$$

$$+2ABm_2[(-Aa^2m_1^2 + ka - Bl_2)\sqrt{l_1} + (-Ba^2m_1^2 + kam_1^2 - Al_1)\sqrt{l_2}$$

$$= 2AB[(Aa^2m_1m_2^2 + kam_1m_2^2 + Bm_1l_2 - Aa^2m_1^2m_2 + kam_2 - Bl_2m_2)\sqrt{l_1}$$

$$+2AB(Ba^2m_1m_2^2 + kam_1^2m_2 + Al_1m_1 - Ba^2m_1^2m_2 + kam_1 - Al_1m_2)\sqrt{l_2}$$

$$= 2AB(\gamma\sqrt{l_1} + \delta\sqrt{l_2})$$

$$[\gamma = (Aa^2m_1m_2^2 + kam_1m_2^2 + Bl_2m_1 - Aa^2m_1^2m_2 + kam_2 - Bl_2m_2)$$

$$\delta = (Ba^2m_1m_2^2 + kam_1^2m_2 + Al_1m_1 - Ba^2m_1^2m_2 + kam_1 - Al_1m_2)]$$

Calculation 7a

$$\alpha = (kAm_1 + A^2am_1 - ABam_2) = A(Aam_1 + km_1 - Bam_2)$$

$$= A[Aam_1 + (Aam_1^2 - Bam_2^2)m_1 - Bam_2]$$

$$= Aa[Am_1 + (Am_1^2 + Bm_2^2)m_1 - Bm_2]$$

$$= Aa[\frac{m_1}{1+m_1^2} + \frac{m_1^3}{1+m_1^2} - \frac{m_1 m_2^2}{1+m_2^2} - \frac{m_2}{1+m_2^2}]$$

$$= Aa[\frac{(m_1 - m_2) + m_1^2(m_1 - m_2)}{(1+m_1^2)(1+m_2^2)}]$$

$$= Aa\frac{(m_1 - m_2)}{(1+m_2^2)} = a\frac{(m_1 - m_2)}{(1+m_1^2)(1+m_2^2)}$$

Calculation 7b

$$\beta = (kBm_2 - B^2 am_2 + ABam_1) = -B(Bam_2 - km_2 - Aam_1)$$

$$= -B[Bam_2 - (Aam_1^2 - Bam_2^2)m_2 - Aam_1]$$

$$= -Ba[Bm_2 - (Am_1^2 - Bm_2^2)m_2 - Am_1]$$

$$= -Ba[\frac{m_2}{1+m_2^2} + \frac{m^3}{1+m_2^2} - \frac{m_1^2 m_2}{1+m_1^2} - \frac{m_1}{1+m_1^2}]$$

$$= -Ba[\frac{(m_2 - m_1) + m_2^2(m_2 - m_1)}{(1+m_1^2)(1+m_2^2)}]$$

$$= -Ba\frac{(m_2 - m_1)}{(1+m_1^2)} = a\frac{(m_1 - m_2)}{(1+m_1^2)(1+m_2^2)}$$

Calculation 8a

$$2AB(Aa^2 m_1 m_2^2 + kam_1 m_2^2 + Bl_2 m_1 - Aa^2 m_1^2 m_2 + kam_2 - Bl_2 m_2) = 2AB\gamma$$

$$(k = Aam_1^2 - Bam_2^2)$$

$$[\gamma = (Aa^2 m_1 m_2^2 + kam_1 m_2^2 + Bl_2 m_1 - Aa^2 m_1^2 m_2 + kam_2 - Bl_2 m_2)$$

$$= [Aa^2 m_1 m_2^2 + (Aam_1^2 - Bam_2^2) am_1 m_2^2 + Bl_2 m_1 - Aa^2 m_1^2 m_2$$

$$+ (Aam_1^2 - Bam_2^2) am_2 - Bl_2 m_2]$$

$$= [Aa^2 m_1 m_2^2 + Aa^2 m_1^3 m_2^2 - Ba^2 m_1 m_2^4 - Ba^2 m_2^3 + Bl_2 (m_1 - m_2)]$$

$$= [\frac{a^2 m_1 m_2^2}{1+m_1^2} + \frac{a^2 m_1^3 m_2^2}{1+m_1^2} - \frac{a^2 m_1 m_2^4}{1+m_2^2} - \frac{a^2 m_2^3}{1+m_2^2} + \frac{l_2 (m_1 - m_2)}{1+m_2^2}]$$

$$= [\frac{a^2 m_1 m_2^2 + a^2 m_1^3 m_2^2 - a^2 m_2^3 - a^2 m_1^2 m_2^3 + l_2 (m_1 - m_2)(1+m_1^2)}{(1+m_1^2)(1+m_2^2)}]$$

$$= [\frac{(m_1 - m_2)(1+m_1^2)(l_2 + a^2 m_2^2)}{(1+m_1^2)(1+m_2^2)}]$$

$$= [\frac{(m_1 - m_2)(1+m_1^2) r^2 (1+m_2^2)}{(1+m_1^2)(1+m_2^2)}] = (m_1 - m_2) r^2$$

$$2AB(Aa^2 m_1 m_2^2 + kam_1 m_2^2 + Bl_2 m_1 - Aa^2 m_1^2 m_2 + kam_2 - Bl_2 m_2) = 2AB\gamma$$

$$= [\frac{2(m_1 - m_2) r^2}{(1+m_1^2)(1+m_2^2)}]$$

Calculation 8b

$$2AB(Ba^2 m_1 m_2^2 + kam_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + kam_1 - Al_1 m_2) = 2AB\delta$$

$$\delta = (Ba^2 m_1 m_2^2 + kam_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + kam_1 - Al_1 m_2)]$$

$$[k = Aam_1^2 - Bam_2^2]$$

$$= [Ba^2 m_1 m_2^2 + (Aam_1^2 - Bam_2^2)am_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + (Aam_1^2 - Bam_2^2)am_1 - Al_1 m_2]$$

$$= [Aa^2 m_1^4 m_2 - Ba^2 m_1^2 m_2^3 - Ba^2 m_1^2 m_2 + Aa^2 m_1^3 + Al_1(m_1 - m_2)]$$

$$= [\frac{a^2 m_1^4 m_2}{1+m_1^2} - \frac{a^2 m_1^2 m_2^3}{1+m_2^2} - \frac{a^2 m_1^2 m_2}{1+m_2^2} + \frac{a^2 m_1^3}{1+m_1^2} + \frac{l_1(m_1 - m_2)}{1+m_1^2}]$$

$$= [\frac{-a^2 m_1^2 m_2^3 - a^2 m_1^2 m_2 + a^2 m_1^3 + a^2 m_1^3 m_2^2 + l_1(m_1 - m_2)(1+m_2^2)}{(1+m_1^2)(1+m_2^2)}]$$

$$= [\frac{a^2 m_1^2 (m_1 - m_2)(1+m_2^2) + l_1(m_1 - m_2)(1+m_2^2)}{(1+m_1^2)(1+m_2^2)}]$$

$$= [\frac{(m_1 - m_2)(1+m_2^2)(a^2 m_1^2 + l_1)}{(1+m_1^2)(1+m_2^2)}]$$

$$= [\frac{(m_1 - m_2)(1+m_2^2) r^2 (1+m_1^2)}{(1+m_1^2)(1+m_2^2)}] = (m_1 - m_2) r^2$$

$$2AB(Ba^2 m_1 m_2^2 + kam_1^2 m_2 + Al_1 m_1 - Ba^2 m_1^2 m_2 + kam_1 - Al_1 m_2) = 2AB\delta$$

$$= \frac{2(m_1 - m_2) r^2}{(1+m_1^2)(1+m_2^2)}$$

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Reference:

[1] A photocopy of the letter from Prof. H.S.M. Coxeter is given below:

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DEPARTMENT OF MATHEMATICS

July 17, 1974.

Mr. D. B. Ghosh,
6 Beniapara Lane,
Serampore,
Dist. Hooghly (West Bengal),
INDIA Pin Code - 712201.

Dear Mr. Ghosh,

Thank you for your letter of the 8th. The Euler-Hazra theorem is quite well known, and easy to prove by the methods of projective geometry; see, for instance, Coxeter & Greitzer, Geometry Revisited (Random House, New York, 1967), pp. 76, 135, 136 and 145 (Theorem 6.51). In your notation, the points of contact of the two tangents from P lie on the polar of P, which is the line QE.

Yours Sincerely,

HSM Coxeter

H.S.M. Coxeter

HSMC:ac

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QUANTUM MECHANICS WITH QUATERNIONIC MASS

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Abstract

Quantum mechanics with quaternionic mass (scalar and vector masses) is considered. The momentum eigen-value equation with quaternionic mass yields the Klein-Gordon equation with a mass consisting of longitudinal and traverse (vector) masses. The scalar field total mass is found to be a sum of these masses. This field appears to be connected with two subfields conserving linear momentum. It is found that a particle with real mass satisfies the quantum Telegraph equation, whereas that one with quaternionic mass satisfies the Klein-Gordon equation. A quantum force acting on the particle is found to be proportional to its velocity. When the particle field is coupled to an electromagnetic field, an additional term in the particle's energy appears reflecting the interaction of the particle's angular momentum with the magnetic field. The quantum particle moves like a fluid whose vorticity (spin) is proportional to the vector mass, $\vec{m}_t$.

Keywords: Quaternionic quantum mechanics; Quaternionic mass; Quantum Telegraph equation; Klein-Gordon equation; De Broglie wave; Euler equation.

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1 Introduction

Maxwell, Planck, de Broglie, and Einstein have laid down the foundational principles of the elementary particles and their fields [1]. In Maxwell's theory, the electric and magnetic fields are treated as one entity if the theory is invariant under the duality transformation. Under such circumstances, Maxwell equations are reduced to two equations. Einstein states that energy and mass are equivalent so that each one can be transformed into another. Lorentz had found that space and time are one entity. They can be transformed into each other by employing Lorentz transformations. Since matter is inscribed into mass and energy into a wave, then matter and wave are equivalent too. This is normally coined in the wave-particle duality. In quaternionic jargon, the quantum equation reflecting the wave-particle duality is the momentum eigen-value equation. This yields a new quantum wave equation exhibiting this duality aspect [2, 3].

The mathematical formalism in which the above principles are realized is via equations of motion. At their highest level, these are the Maxwell, Klein-Gordon, and Dirac equations. Maxwell equations describe pure wave phenomena, and Klein-Gordon and Dirac describe the particle phenomena. However an equation that describes the evolutionary state is found in [2, 3]. The two forms of existence are described by fields. Any change in this field will be exhibited as a wave. Besides the dynamical evolution of a quantum object, an aggregation of particles in an ensemble is found to be governed by specific statical laws. Further, it is found that the dynamics and statistics are connected. Fermions that are governed by the Dirac equation follow different statistics than those of bosons which are described by the Klein-Gordon equation. Besides, it is also found that Maxwell equations govern other kinds of bosons with different statistics. In effect, all particles find their way to be described by appropriate dynamical as well as statistical equations.

Objects like molecules (atoms) follow other statistics known as the Maxwell-Boltzmann statistics. Therefore, the physical properties of microparticles are intimately connected. It is well known that all objects (particles) have inertia expressing the extent to which they resist a change of state (motion to rest or rest to motion). Inertia can be modeled as some kind of friction.

In quantum mechanics, a quantum particle is modeled by a wave packet in such a way that its group velocity is equal to the particle velocity. However, the phase velocity of a single wave is different from the particle and group velocities. In our anticipated description, we think of a particle and its wave as two manifestations of one object. When the particle behaves as a wave its momentum is equal to that when it behaves like a normal particle. The wave has a vector mass whereas the particle has a scalar mass. The two entities are connected, however.

Space can not be compatible with the notion of inertia, however. A fluid (ether) filling the whole space was promoted by physicists but later rejected by the famous Michelson–Morley experiment. The experiment did not rule out the presence of the fluid but rather excluded its existence to carry the electromagnetic wave. The electromagnetic wave was thought of as a mechanical wave that requires a priori a medium to propagate in. The modern understating of the space, today, allows some kind of minimal field to be pervading the whole space. This expresses a background due to quantum fluctuations in the field. It is called the *vacuum* that today bears a physical existence.

Mass is a property that distinguishes an object. The wave inertia however is not characterized by its mass. Therefore, the notion of mass has to be extended to incorporate particles as well as waves. There exists now a need to distinguish different kinds of mass that could be due to the field of force under which the object interacts. With the same token, an object can have two kinds of charge too playing analogous roles. One mass (charge) represents the matter nature while the other one characterizes the wave nature. A mass besides being the source of matter-wave could also act as a source for the electromagnetic field.

We have found that Quaternions are the appropriate mathematical tools that can deal with all of the above identification of the object. This is owing to the spectacular algebra of Quaternions. Treating the particle mass as quaternion yields very interesting results. Since a quaternion consists of scalar and vector components, a quaternionic mass embodies the particle (scalar) and wave (vector) aspects. In the de Broglie theory, a photon wave, with zero mass, is used to draw a parallel with that one having a mass.

In the present formalism, we however employ a different prescription.

We would like to explore these possibilities by applying the quaternionic mass in Dirac formalism to manifest this. This recipe turned out to be equivalent to introducing an interaction of the quantum particle with an external field. Previously, the normal mechanism to do so is via the minimal coupling method.

This paper is scheduled as follows: we introduce in Section 2 quaternionic quantum mechanics with real and quaternionic mass. In this section, we derived the energy conservation (continuity) equation of the quantum system. We then showed that the duality aspect is guaranteed by deriving a particle-wave momentum equation connecting the wave momentum to that of the particle. This relationship reveals that a quaternionic mass representation of a quantum particle realizes the duality principle due to de Broglie. While the ordinary quaternionic momentum eigen-value equation with real mass yields a Telegraph equation, that one with quaternionic mass yields the Klein-Gordon equation for a field whose mass consists of two masses. In Section 3 we study the interaction of a quantum particle with an electromagnetic field. An interaction term showing the coupling of a particle's angular momentum with the magnetic field appears in the particle's Hamiltonian. A Lorentz-like force deemed to govern the quantum particle motion is presented in Section 4. We end our paper by concluding remarks in Section 5.

2 Quantum mechanics with quaternionic mass

The Dirac (momentum eigen-value) equation with real mass, m, was studied before yielding [2, 3]

$$\frac{1}{c^2}\frac{\partial \psi_0}{\partial t} + \vec{\nabla}\cdot\vec{\psi} + \frac{m}{\hbar}\psi_0 = 0\,, \tag{1}$$

$$\frac{\partial \vec{\psi}}{\partial t} + \vec{\nabla}\psi_0 + \frac{mc^2}{\hbar}\vec{\psi} = 0\,, \tag{2}$$

$$\vec{\nabla}\times\vec{\psi} = 0\,. \tag{3}$$

Equations (1) - (3) yield the wave equations

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2}-\nabla^2\psi_0+\frac{2m}{\hbar}\frac{\partial\psi_0}{\partial t}+\left(\frac{mc}{\hbar}\right)^2\psi_0=0\,,\ \frac{1}{c^2}\frac{\partial^2\vec{\psi}}{\partial t^2}-\nabla^2\vec{\psi}+\frac{2m}{\hbar}\frac{\partial\vec{\psi}}{\partial t}+\left(\frac{mc}{\hbar}\right)^2\vec{\psi}=0\,. \tag{4}$$

These are the undistorted Telegraph equations that are known to describe the propagation of the electric current and voltage in transmission lines. Interestingly, the inertial (quantum) and electromagnetic waves of the electron are coherent. The solution of the above quantum Telegraph equation is a wavepacket consisting of waves traveling to the left and right with the speed of light. This is an advantage of the quantum Telegraph equation over the Schrodinger equation where the solution of the latter is a plane wave. For the Schrodinger equation to describe the motion of a particle two plane waves have to be added to obtain a wavepacket solution.

We would like here to entertain the possibility of studying the momentum eigen-value equation with quaternionic mass. To this aim let us write the quaternionic momentum eigen-value generalizing the ordinary one. It is expressed in the form [2, 3]

$$\tilde{P}\,\tilde{\Psi}^*=c\tilde{M}\,\tilde{\Psi}^*\,,\tilde{P}=\left(\frac{i}{c}\hat{E}\,,\hat{\vec{p}}\right),\tilde{\Psi}=\left(\frac{i}{c}\psi_0\,,\vec{\psi}\right),\tilde{M}=(im_\ell\,,\vec{m}_t\,)\,. \tag{5}$$

Here, $\hat{E}$ and $\hat{\vec{p}}$ are the energy and momentum operators. The two masses m_ℓ and $\vec{m}_t$ are the longitudinal and transverse masses describing the wave-particle duality. Here $\tilde{\Psi}$ describes the state of the quantum particle consisting of a vector and a scalar part that is analogous to the scalar and vector potentials describing the electromagnetic field. The quaternion mass, $\tilde{M}$, can be thought of as a mass field. The particle interacts with the outside world by coupling this mass field with the other fields in which the particle exists. The physical meaning of the masses, $\vec{m}_t$ and m_ℓ will be given later on.

Applying the quaternionic product rule, and equating the real and imaginary parts in the two sides of the resulting equations, one finds [4]

$$\frac{1}{c^2}\frac{\partial\psi_0}{\partial t}+\vec{\nabla}\cdot\vec{\psi}=0\,, \tag{6}$$

$$\frac{\partial \vec{\psi}}{\partial t} + \vec{\nabla}\psi_0 + \frac{c^2}{\hbar}\vec{m}_t \times \vec{\psi} = 0\,, \tag{7}$$

$$\vec{\nabla} \times \vec{\psi} + \hbar^{-1}(m_\ell c\,\vec{\psi} - \vec{m}_t\psi_0) = 0\,, \tag{8}$$

and

$$m_\ell\psi_0 - c\,\vec{m}_t \cdot \vec{\psi} = 0\,. \tag{9}$$

Now equations (7) and (9) yield

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0\,, \qquad \vec{J} = c\,\vec{m}_t\psi_0\,, \qquad \rho = m_\ell\psi_0\,. \tag{10}$$

Thus if $\vec{J} = \rho\,\vec{v}$, then

$$m_\ell\vec{v} = c\,\vec{m}_t\,. \tag{11}$$

Equation (11) can be seen as a momentum conservation of the two masses representing the quantum particle. Since in de Broglie's theory a particle has a wave nature, the term $c\,\vec{m}_t$ in eq.(11) could account for the momentum of the wave associated with the particle. If we say that the matter-wave momentum is $\vec{p}_W = c\vec{m}_t$, then using Einstein's mass-energy relationship, $E = mc^2$, one finds the value of the matter-wave momentum to be $p_W = E/c$. This agrees with the standard representation of the momentum of light (photon). The vector mass, $\vec{m}_t$, may point along the direction in which matter-wave travels.

Recall that when a charged particle interacts with an electromagnetic field, the field imparts a momentum $\vec{p}_A = q\vec{A}$ such that the total momentum the charge receives is $\vec{p}' = \vec{p} + q\vec{A}$. This allows us to deduce that the momentum provided to the particle is $\vec{p}_f = c\,\vec{m}_t$. When a charged particle interacts with an electromagnetic field then, $q\vec{A} = c\,\vec{m}_t$. Hence, the transverse (vector) mass describes the momentum that a particle receives while moving. If the particle moves freely, $\vec{m}_t = 0$. This urges us to define m_ℓ as the mass of the free particle. It is thus very intriguing to encapsulate the two masses in a single representation, the quaternionic mass.

Now eqs.(6) and (7) yield

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0\,, \qquad \vec{S} = \psi_0\vec{\psi}\,, \qquad u = \frac{\psi^2}{2} + \frac{\psi_0^2}{2c^2}\,, \tag{12}$$

which represents the energy conservation equation of the system whose mass in quaternionic. The energy conservation equation associated with real particle mass, eqs.(1) - (3), is given by

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\frac{u}{\tau}, \qquad \tau = \frac{\hbar}{2mc^2}, \tag{12a}$$

where u represents the energy density of the particle field and $\vec{S}$ is its energy flux. It was Maxwell who thought of electromagnetic field (wave) propagation as fluid flow. It is interesting to observe that the introduction of the quaternionic mass removes the dissipation in the system that is associated with eqs.(1) - (3).

Multiply eq.(7) by ψ_0 and employ eqs.(6) and (8) to get the momentum (force) equation

$$\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} + (\vec{\psi}\vec{\nabla} \cdot \vec{\psi} + \vec{\psi} \cdot \vec{\nabla}\vec{\psi}) + \vec{\nabla}\left(\frac{\psi_0^2}{2c^2} - \frac{\psi^2}{2}\right) = 0, \tag{13}$$

In components form, eq.(13) can be expressed as

$$\frac{1}{c^2}\frac{\partial S_i}{\partial t} + \partial_j(\psi_i\psi_j) + \partial_i\left(\frac{\psi_0^2}{2c^2} - \frac{\psi^2}{2}\right) = 0, \tag{14}$$

which can be written as

$$\frac{1}{c^2}\frac{\partial S_i}{\partial t} + \partial_j\left(\psi_i\psi_j + \delta_{ij}(\frac{\psi_0^2}{2c^2} - \frac{\psi^2}{2})\right) = 0, \qquad -\frac{\partial g_i}{\partial t} + \partial_j\sigma_{ij} = 0, \tag{15}$$

where

$$\sigma_{ij} = -\psi_i\psi_j - \delta_{ij}\left(\frac{\psi_0^2}{2c^2} - \frac{\psi^2}{2}\right), \tag{16}$$

is a symmetric stress-tensor of the matter field. Interestingly, eq.(15) is the momentum conservation equation that is analogous to the electromagnetic one, where the momentum of the matter wave is defined by $\vec{g} = \frac{\vec{S}}{c^2}$ as well. The quantum particle (de Broglie wave) travels in space like the fluid flow. Thus, eq.(15) describes the force density that the matter wave impinges.

The momentum conservation equation of a quantum particle described by its real mass is given by

$$-\frac{\partial g_i}{\partial t} + \partial_j \sigma_{ij} = \frac{g_i}{\tau}\,. \tag{15a}$$

Eqs.(6) - (9) can be shown to represent a wave equation (matter wave) [4]

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2} - \nabla^2\psi_0 + \left(\frac{Mc}{\hbar}\right)^2 \psi_0 = 0, \qquad \frac{1}{c^2}\frac{\partial^2\vec{\psi}}{\partial t^2} - \nabla^2\vec{\psi} + \left(\frac{Mc}{\hbar}\right)^2 \vec{\psi} = 0\,, \tag{17}$$

where

$$M^2 = m_t^2 - m_\ell^2\,. \tag{18}$$

Equation (17) is but the Klein-Gordon equation of a scalar field whose effective mass is M which I prefer to call the invariant mass. It is thus interesting that the momentum eigen-value equation with a quaternion mass gives the Klein-Gordon equation. It seems that the Klein-Gordon field is a superposition of two kinds of fields. A tachyonic Klein-Gordon equation results when $|m_t| < m_\ell$. As evident from eq.(18), if the scalar field has the same mass components, it then behaves like a massless field (photon). It is quite remarkable to see that a particle described by a real mass satisfies the quantum Telegraph equation while that one described by a quaternionic mass yields the Klein-Gordon equation. Note that the Dirac equation can be obtained from the quantum Telegraph equation by taking the mass to be imaginary [2].

The quaternionic momentum eigen-value equation associated with real mass was found to lead to the Telegraph equation (or a dissipative Klein-Gordon equation) [5]. The dissipative term is removed when quaternionic mass is proposed instead.

Applying eq.(11) in eq.(18) yields

$$m_\ell = \frac{M}{\sqrt{1-\frac{v^2}{c^2}}}\,, \qquad \vec{m}_t = \left(\frac{\vec{v}}{c}\right) m_\ell\,. \tag{19}$$

The longitudinal mass m_ℓ is but the relativistic mass and M is the field rest-mass. Notice that in general the value of the transverse mass is smaller

than the longitudinal one. We could now say that the transverse mass is the *active mass* whereas the longitudinal mass is a *passive mass* of the particle. When the particle is not interacting its transverse vanishes and its longitudinal mass remains. This new concept of masses should find application in scattering problems of microscopic particles having particle and wave natures. In such cases, the conservation of energy and momentum should be taken in the association of a quaternionic perspective.

Interestingly, Eq.(19) associates inertia with motion. In the ordinary Newton's laws, the inertia of a macroscopic object is connected with its acceleration. However, in quantum mechanics the force is directly proportional to velocity [6]. The second equation in eq.(19) can be expressed as the Einstein's relativistic equation between velocity, momentum and energy that is valid for matter as well as waves, $\vec{v} = (c^2/E)\,\vec{p}$, upon expressing it as $\vec{v} = [c^2/(m_\ell c^2)]\vec{m}_t c$, with $E = m_\ell c^2$ and $\vec{p} = \vec{m}_t c$. The latter momentum can be seen as a momentum of the wave-packet mimicking the particle. Hence, this analogy provides a faithful realization of our theory. Thus, for particle interactions the momentum and energy conservations are conservations of the vector mass and relativistic masses, $\vec{m}_t$ and m_ℓ, respectively. The relation in eq.(19) was already derived in the context of a different eigen-value equation of a quantum particle [7].

Defining $\rho = \psi_0/c^2$ and $\vec{J} = \vec{\psi}$, equations (6) and (7) can be seen as the continuity and momentum conservation equations governing matter flow. It is valid for mass as well as charge flow. It therefore fully characterizes the moving quantum particle. For mass (fluid) motion, eq.(7) becomes

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}(c^2\rho) - \frac{c^2\rho}{\hbar}\vec{m}_t \times \vec{v}\,, \tag{7a}$$

that is a force equation of a fluid with pressure $p = c^2\rho$ in addition to a transverse force perpendicular to the fluid velocity. The above equation can be expressed as

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\vec{\nabla}p + \vec{v} \times (\vec{\nabla} \times \vec{v})\,. \tag{7b}$$

The last term in eq.(7b) suggests a vorticity (spin) given by $\vec{\omega} = \vec{\nabla} \times \vec{v} = \frac{c^2\vec{m}_t}{\hbar}$ [8]. The appearance of the Planck constant in the vorticity implies it

is quantum. It is remarkable that the vector mass, $\vec{m}$, is connected to the particle's spin.

3 Interacting quantum particle with an electromagnetic field

The interaction of a charged (q) quantum particle with an electromagnetic field can be found by using the minimal coupling recipe, where the particle momentum and energy are replaced, respectively, by, $\vec{p} \rightarrow \vec{p} - q\vec{A}$ and $E \rightarrow E - q\varphi$ [1]. This is found to alter eqs.(1) - (3) to read [9]

$$\frac{1}{c^2}\frac{\partial \psi_0}{\partial t} + \vec{\nabla}\cdot\vec{\psi} + \frac{m}{\hbar}\psi_0 = 0\,, \tag{20}$$

$$\frac{\partial \vec{\psi}}{\partial t} + \vec{\nabla}\psi_0 + \frac{mc^2}{\hbar}\vec{\psi} + \frac{cq}{\hbar}\vec{A}\times\vec{\psi} = 0\,, \tag{21}$$

$$\vec{\nabla}\times\vec{\psi} + \hbar^{-1}(c^{-1}q\varphi\,\vec{\psi} - c^{-1}q\vec{A}\psi_0) = 0\,, \tag{22}$$

and

$$c^{-2}q\varphi\,\psi_0 - q\vec{A}\cdot\vec{\psi} = 0\,, \tag{23}$$

where φ and $\vec{A}$ are the scalar and vector potential of the electromagnetic field. An interesting analogy between eqs.(20) - (23) and eqs.(6) - (9) can be drawn. This is as follows: $m = 0$, $c\,\vec{m}_t \rightarrow q\vec{A}$, and $m_\ell c^2 \rightarrow q\,\varphi$. Therefore, introducing a quaternionic mass in a theory is equivalent to dealing with interaction of a particle with an external field.

Equations (20) - (23) reveal that ψ_0 satisfies the Telegraph's wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2} - \nabla^2\psi_0 + \frac{2m}{\hbar}\frac{\partial\psi_0}{\partial t} + \left(\frac{m^2c^2}{\hbar^2} + \frac{q^2}{\hbar^2}(A^2 - \frac{\varphi^2}{c^2})\right)\psi_0 = \frac{cq}{\hbar}\vec{\psi}\cdot\vec{B}\,, \tag{24}$$

and

$$\frac{1}{c^2}\frac{\partial^2\vec{\psi}}{\partial t^2} - \nabla^2\vec{\psi} + \frac{2m}{\hbar}\frac{\partial\vec{\psi}}{\partial t} + \left(\frac{m^2c^2}{\hbar^2} + \frac{q^2}{\hbar^2}(A^2 - \frac{\varphi^2}{c^2})\right)\vec{\psi} = \frac{q}{c\hbar}(\vec{E}\times\vec{\psi} + \psi_0\vec{B})\,, \tag{25}$$

with variable mass and source terms that couple to the magnetic field. It is intriguing that the above two equations are coupled to each other. The right-hand sides of eqs.(24) and (25) shows that the electromagnetic field coupled to the particle's fields (wavefunctions), ψ_0 and $\vec{\psi}$. For homogenous wave equations, eqs.(24) and (25) imply that

$$\vec{\psi} \cdot \vec{B} = 0\,, \qquad \vec{B} = \frac{\vec{\psi}}{\psi_0} \times \vec{E}\,. \tag{24a}$$

These show that the electromagnetic field is perpendicular to the particle's vector wavefunction, $\vec{\psi}$. Equation (24a) reduces to the electrodynamics of a moving charge for, $\vec{v}\psi_0 = c^2\vec{\psi}$. With the conditions in eq.(24a), eqs.(24) and (25) decouple.

It would be interesting if we connect the vector wavefunction to the angular momentum, $\vec{L}$, of the particle by

$$c\hbar\vec{\psi} = \vec{L}\psi_0\,, \tag{26}$$

so that the right-hand side of eq.(25) will describe a particle's orbital interaction with the magnetic field of the form

$$\frac{1}{c^2}\frac{\partial^2\psi_0}{\partial t^2} - \nabla^2\psi_0 + \frac{2m}{\hbar}\frac{\partial\psi_0}{\partial t} + \left(\frac{m^2c^2}{\hbar^2} + \frac{q^2A^2}{\hbar^2} - \frac{q^2\varphi^2}{\hbar^2c^2} - \frac{q}{\hbar^2}\vec{L}\cdot\vec{B}\right)\psi_0 = 0\,. \tag{27}$$

Equation (27) can be expressed in an energy equation as

$$\frac{\hbar^2}{2mc^2}\frac{\partial^2\psi_0}{\partial t^2} - \frac{\hbar^2}{2m}\nabla^2\psi_0 + \hbar\frac{\partial\psi_0}{\partial t} + \left(\frac{mc^2}{2} + \frac{q^2A^2}{2m} - \frac{q^2\varphi^2}{2mc^2} - \frac{q}{2m}\vec{L}\cdot\vec{B}\right)\psi_0 = 0\,. \tag{28}$$

Under the rotation of space and time coordinates, *i.e.*, $-\pi/2$ ($t \to -it$ and $r \to -ir$)[10], eq.(28) becomes

$$i\hbar\frac{\partial\psi_0}{\partial t} = \frac{\hbar^2}{2mc^2}\frac{\partial^2\psi_0}{\partial t^2} - \frac{\hbar^2}{2m}\nabla^2\psi_0 - \left(\frac{mc^2}{2} + \frac{q^2A^2}{2m} - \frac{q^2\varphi^2}{2mc^2} - \frac{q}{2m}\vec{L}\cdot\vec{B}\right)\psi_0\,. \tag{28a}$$

The last term in the left-hand side of eq.(28a) embodies the interaction of the particle's angular momentum with the external magnetic field. The

first term in eq.(28a) will have a small contribution to the low-frequency limit. For such a case eq.(28a) approximates to Schrodinger equation. If we chose in eq.(26) the spin angular momentum instead of the orbital angular momentum, then $c\hbar\vec{\psi} = 2\vec{S}\psi_0$, where $2\vec{S} = \vec{\sigma}$ are Pauli's matrices, then the energy term in eq.(28a) will describe the particle's spin angular momentum. Notice that the vector wavefunction equation, eq.(25), doesn't involve such a term.

For massive electrodynamics, one multiplies eq.(25) by $\frac{\varepsilon_0 c\hbar}{q}$ and choose, $\vec{\psi} = \vec{A}$ and $\psi_0 = \varphi$, so that eq.(25) becomes

$$\frac{1}{c^2}\frac{\partial^2\vec{Y}}{\partial t^2} - \nabla^2\vec{Y} + \frac{2m}{\hbar}\frac{\partial\vec{Y}}{\partial t} + \left(\frac{m^2c^2}{\hbar^2} + \frac{q^2}{\hbar^2}(A^2 - \frac{\varphi^2}{c^2})\right)\vec{Y} = \vec{s}, \qquad (29)$$

where

$$\vec{s} = \varepsilon_0(\vec{E}\times\vec{A} + \varphi\vec{B}), \qquad \vec{Y} = \frac{\varepsilon_0 c\hbar}{q}\vec{A}. \qquad (30)$$

The quantity, $\vec{s}$, is the spin density of the electromagnetic field and the field $\vec{Y}$ is a linear momentum vector field [11]. This latter momentum is new to electrodynamics. Equation (29) shows that the spin $\vec{s}$ is the source for the momentum vector field, $\vec{Y}$. Equation (27) will yield a nonlinear Telegraph's equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi + \frac{2m}{\hbar}\frac{\partial\varphi}{\partial t} + \left(\frac{m^2c^2}{\hbar^2} + \frac{q^2A^2}{\hbar^2} - \frac{q}{\hbar^2}\vec{L}\cdot\vec{B}\right)\varphi - \frac{q^2}{\hbar^2c^2}\varphi^3 = 0. \quad (31)$$

Equation (31) can be expressed as an energy equation as

$$\frac{\hbar^2}{2mc^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\hbar^2}{2m}\nabla^2\varphi + \hbar\frac{\partial\varphi}{\partial t} + \left(\frac{mc^2}{2} + \frac{q^2A^2}{2m} - \frac{q}{2m}\vec{L}\cdot\vec{B}\right)\varphi - \frac{q^2}{2mc^2}\varphi^3 = 0. \qquad (32)$$

Upon space and time rotation by $-\pi/2$ ($t \to -it$ and $r \to -ir$) [10], eq.(32) yields

$$i\hbar\frac{\partial\varphi}{\partial t} = \frac{\hbar^2}{2mc^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\hbar^2}{2m}\nabla^2\varphi - \left(\frac{mc^2}{2} + \frac{q^2A^2}{2m} - \frac{q}{2m}\vec{L}\cdot\vec{B}\right)\varphi + \frac{q^2}{2mc^2}\varphi^3. \qquad (33)$$

Equation (33) can be seen as a late-time nonlinear Schrodinger equation of a charged particle interacting with an electromagnetic field with φ as a wavefunction. The fifth term in the left-hand side of eq.(33) would represent the interaction orbital angular momentum with the magnetic field, where the particle spin $\vec{S} = 0$.

4 Extended Lorentz force

The Lorentz force describing the force acting of the quantum particle can be extended to read

$$\tilde{F} = c\tilde{M}^*\tilde{V}\tilde{\nabla}\tilde{\Psi} = \left(\frac{i}{c}P, \vec{f}\right), \qquad \tilde{V} = (ic, \vec{v}), \tag{34}$$

where $\tilde{\Psi}$ is considered to be dimensionless. Using the quaternion product rule mentioned above and equating the real and imaginary components in the two sides of the resulting equation, yield

$$\vec{f} = \frac{M^2c^3}{\hbar}\vec{\psi} - M\frac{\psi_0}{c}\frac{\vec{v}}{\tau}, \; P = \frac{Mc}{\tau}(\psi_0 - \vec{v}\cdot\vec{\psi}), \; \vec{m}_t\cdot\vec{v} = m_\ell c, \; \tau = \frac{\hbar}{Mc^2}, \tag{35}$$

upon employing eqs.(6) - (9). The first term in the force is analogous to the electric force and the second one to the friction force that occurs in a conductor due to scattering between electrons. It is also like a viscous force that requires the presence of a fluid that permeates the whole space. The time τ is a characteristic time reflecting the jittery (trembling) motion of a quantum particle as observed by Schrodinger for the electron free-motion [12]. It seems a quantum particle jitters because it experiences a quantum (viscous) force as it travels in space. The jitter frequency here is twice that due to the electron in the Dirac theory. The quantum resistive force may explain the origin of inertia that any particle experiences as it starts to move.

5 Concluding remarks

We have extended in this work our earlier work on quantum mechanics where the mass is now treated as a quaternion. The quaternionic momen-

tum eigen-value equation leads to a Telegraph equation when a real mass is used, it however leads to the Klein-Gordon equation when a quaternionic mass is used instead. This scalar field has an effective mass that is a combination of its two masses. The vector and scalar masses connect the particle matter-wave duality where the wave's momentum is equal to the particle's momentum. This is expressed by the equation, $\vec{m}_t c = m_0 \vec{v}$, where $\vec{v}$ is the particle's velocity. When the quantum particle interacts with an electromagnetic field, the particle's angular momentum is coupled to the magnetic field. A quantum frictional force acting on the moving particle is found to be proportional to its velocity. The interaction of the quantum particle with the electromagnetic field can be obtained from that of a free particle with quaternionic mass provided one defines $c\vec{m}_t \equiv q\vec{A}$ and $m_\ell c^2 \equiv q\varphi$. A new linear momentum field is found to be associated with the spin of the electromagnetic field. A charged particle interacting with an electromagnetic field is found to be governed by a Schrodinger-like equation where the particle spin (orbital) angular momentum is coupled to the electromagnetic field.

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CHAOTIC DYNAMICS ON THE SPHERE

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Abstract

First, we construct the image of the torus on the two-layer shell of the sphere and note that the isometries of the image of the torus on the sphere generate the unitary group $U(2)$, and then we establish that, as a result of the action of the modular group on the sphere, of all closed torus windings wound around the sphere, singly wound windings are allocated, which are indexed by a set of primes. Next, we form a representation of the sphere winding for the Jacobi theta function and the Riemann zeta function, and then, considering the chaotic dynamics on the sphere, we notice that in the problem of random walk along the broken lines of the winding of the sphere, the concept of a complex probability amplitude arises quite naturally, and the dynamics of the probability amplitude of a wandering particle obeys a differential equation generalizing the Schrödinger equation.

This paper pursues the task of searching for a geometric and at the same time a dynamic model of an electron.

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1 Introduction

There are two main directions of electron modeling in physics. On the one hand, this is a set of toroidal ring models, which were initiated by Parson [1], and on the other hand, this is the standard model of electroweak interactions, which originates from the work of Yang, Mills [2], then was continued by Glashow [3] and completed by Weinberg [4] and Salam [5]. However, despite the successful description of the physical properties of the electron in the framework of quantum mechanics and quantum field theory, such mysterious properties as the origin of the group of gauge symmetries of electroweak interactions and the origin of the electron wave function remain misunderstood until now. And since these riddles relate to the questions of the foundation of physics, they should be solved through mathematical modeling, but with mandatory subsequent verification of compliance with physical laws. In this paper, a toroidal spherical model is presented, from the geometry of which the group $U(1) \times SU(2)$ is extracted, and the one-dimensional Schrödinger equation is derived from the dynamics of a random walk on the winding of a sphere. At the same time, it should be clearly understood that 'dynamics on the sphere' is not physics (and not even an alternative to the electroweak model), but only mathematics, more precisely, a mathematical example of one dynamic model. It is still difficult to say what relation our toroidal spherical model has to the real Universe, but the author is sure [6] that dynamic flows on a seven-dimensional sphere are the real Universe.

In the second and third sections of the paper, we prepare a mathematical tool, which we then use in the fourth section to study the dynamic model of random walk along the turns of the toroidal winding of a sphere. Note that, since even the irrational windings of the torus are infinite, the compactness of the toroidal spherical model does not contradict the infinity of space-time coordinates. In addition, we believe that in unobservable extra dimensions, the electron has a toroidal spherical shape, and this does not contradict the point-like nature of the electron in the observed Euclidean space.

2 The modular group action on the sphere

For explanations of basic algebra-geometric concepts, you can refer to [7]. Here we will take a Cartesian plane (x, y) with perpendicular (in the Euclidean sense) coordinate axes and construct from it a complex plane $z = x + \mathrm{i}y$. If you combine the axes of a Cartesian plane (respectively the real and imaginary axis of the complex plane) with isotropic coordinates in the pseudo-Euclidean plane with orthonormal coordinates $(\frac{x+y}{2}, \frac{x-y}{2})$, the square pseudo-Euclidean length of vector $\xi = (x, y)$ will be equal to the product of xy, and the hyperbolic angle of this vector to the corresponding axis coordinates of the pseudo-Euclidean plane is equal to $-\frac{1}{2}\log\left|\frac{x}{y}\right|$. In this case, the usual (Euclidean) angle of the vector $\xi = (x, y)$ to the isotropic line x is $\arctan\left(\frac{y}{x}\right)$.

Factorizing the coordinates of the Cartesian plane $z = x + \mathrm{i}y$ modulo one

$$x' \equiv x \pmod 1 \qquad y' \equiv y \pmod 1 \tag{1}$$

turns it into a torus $S^1 \times S^1 \cong \left(\mathrm{e}^{2\pi \mathrm{i}x}, \mathrm{e}^{2\pi \mathrm{i}y}\right)$, and if you require saving the equivalence relation 1 (which is equivalent to saving on the torus of the image of integer lattice nodes), then the group of admissible linear transformations of the Cartesian plane narrows to $SL(2, \mathbb{Z})$. In addition, if we neglect the orientation of lines (and their windings on the torus) passing through the reference point of the Cartesian plane, the group of admissible transformations will narrow to the modular group $SL(2, \mathbb{Z})/\{\pm \mathrm{I}\}$. It should also be recalled that mapping a complex plane onto a complex plane with a punctured point

$$\mathbb{C} \to \mathbb{C} \setminus \{0\}: \quad x + \mathrm{i}y \to \mathrm{e}^{2\pi \mathrm{i}(x+\mathrm{i}y)} \tag{2}$$

folds the complex plane into a tube (infinite cylinder) and turns this tube onto a plane with a punctured point, and mapping

$$\mathbb{R} \to \mathbb{C} \setminus \{0\}: \quad t \to \mathrm{e}^{2\pi \mathrm{i}t(a+\mathrm{i}b)} \tag{3}$$

where $a, b \in \mathbb{R}$, draws a spiral Sp on the complex plane.

Consider also the factorization modulo 2 of the product of coordinates of vectors in the Cartesian plane lying on the straight line $y = \tau x$, where

$\tau > 0$,

$$(x, y = \tau x) : x'y' \equiv xy = \tau x^2 \pmod 2 \tag{4}$$

In other words, using the pseudo-Euclidean metric of the complex plane, we construct a mapping of the winding onto the unit complex double argument circle of the complex straight line, which is equivalent to the timelike straight line of the pseudo-Euclidean plane

$$(x + \mathrm{i}\tau x) \to 2S^1 : \quad (x + \mathrm{i}\tau x) \to \mathrm{e}^{2\pi\mathrm{i}\sqrt{x\tau x}} = \mathrm{e}^{\pi\mathrm{i}\tau x^2} \tag{5}$$

Finally, let us define the mapping of the winding of a complex straight line onto a complex double argument spiral, consistent with the winding of a straight line onto a circle

$$(rx + \mathrm{i}\tau x) \to 2Sp : \quad (rx + \mathrm{i}\tau x) \to \mathrm{e}^{\pi(r-1)x^2}\mathrm{e}^{\pi\mathrm{i}\tau x^2} \tag{6}$$

We now proceed to describe the mapping of the torus $S^1 \times S^1$ onto the two-layer shell of a sphere without polar caps $S^2 \setminus \{\mathrm{S}, \mathrm{N}\}$. To do this, on the Arctic Circle, which has a latitude of ϑ, mark the point O_N, which has a longitude of λ, and on the South polar circle, which has a latitude of $-\vartheta$, mark the point O_S at longitude $\lambda + \pi$. Next, we will construct on the sphere the image of the Villarceau circle, that is, draw through the points O_N, O_S the great circle of the sphere so that it touches (but does not intersect) the corresponding polar circle at each of these points. Then the semicircle $]O_S, O_N[$ with coordinates $\varphi(]O_S, O_N[) =]0, \pi[$ lies in one (for example, outer) layer, and the semicircle $]O_N, O_S[$ with coordinates $\varphi(]O_N, O_S[) =]\pi, 2\pi[$ lies in another (inner) the shell layer of the sphere $S^2 \setminus \{\mathrm{S}, \mathrm{N}\}$, while the points (O_S, O_N) belong to two layers at the same time. So, as a result, we got the desired image of the torus on the sphere

$$\left(\vartheta = \frac{\pi}{4}, \quad \lambda = |\pi x| \bmod 2\pi, \quad \varphi = |\pi\sqrt{xy}| \bmod \pi\right) \tag{7}$$

Note here that the proper isometries of the torus and the rotation of the torus image on the sphere about the axis (O_S, O_N) through the angle ϕ generate the unitary group $U(2)$, and the motion of the torus along circles Villarceau and rotation about the (O_S, O_N) axis generate the special unitary group $SU(2)$. Indeed, if $O, O' \in SO(2)$ and $T \in \mathrm{diag}\left[\mathrm{e}^{\mathrm{i}\varphi_1}, \mathrm{e}^{\mathrm{i}\varphi_2}\right]$, then

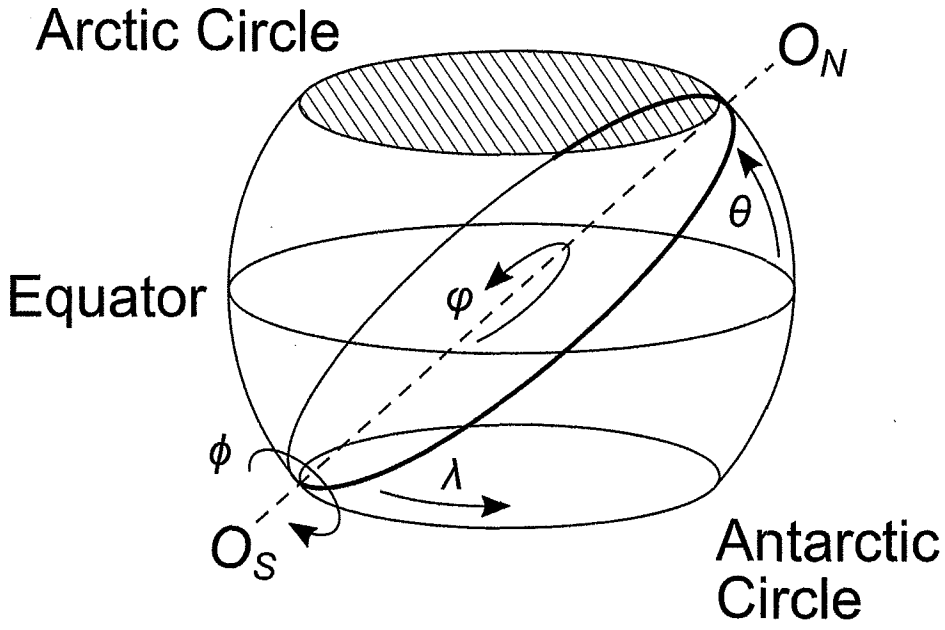

Figure 1: The image of a torus on a two-layer shell of a sphere

$OTO' \in U(2)$, and if $T \in \mathrm{diag}\left[\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}(2\pi-\varphi)}\right]$, then $OTO' \in SU(2)$, and in addition, the sum of the dimensions of the group of diagonal matrices and two orthogonal groups is equal to the dimension of the corresponding unitary group.

Let us now see how the angular values ϑ, φ from the winding mapping 7 are transformed with a hyperbolic rotation of the pseudo-Euclidean plane. Let a hyperbolic rotation $z' = x' + \mathrm{i}y' = \mathrm{e}^{\pm\psi}(x + \mathrm{i}y) = \mathrm{e}^{\psi}x + \mathrm{i}\mathrm{e}^{-\psi}y$. Then, for the vector ξ whose image lies on the Villarceau circle (and by virtue of which $\xi_x = \xi_y$), the identities $\xi'_x\xi'_y = \mathrm{e}^{\psi}\xi_x\mathrm{e}^{-\psi}\xi_y = \xi_x\xi_y$, $\frac{\xi'_x}{\xi'_y} = \mathrm{e}^{2\psi}\frac{\xi_x}{\xi_y} = \mathrm{e}^{2\psi}$, we get transformations

$$\begin{aligned}(x + \mathrm{i}y) \to \quad & \mathrm{e}^{\pm\psi}(x + \mathrm{i}y) = (x' + \mathrm{i}y') \quad \to \\ & \left(\vartheta' = \arctan\left(\mathrm{e}^{-2\psi}\right),\ \varphi' = \left|\sqrt{\xi_x\xi_y}\right| \bmod \pi\right)\end{aligned} \tag{8}$$

In turn, under the Euclidean rotation of isotropic straight lines in the pseudo-Euclidean plane, $z' = x' + \mathrm{i}y' = \mathrm{e}^{\mathrm{i}\phi}(x + \mathrm{i}y)$ the coordinates of the vector ξ are preserved, that is, $\xi'_x = \xi_x$, $\xi'_y = \xi_y$, so the winding mapping ref sphere1 just add a Euclidean rotation of the shell around the (O_S, O_N) axis. In this case, we get a chain of transformations

$$\begin{aligned}(x + \mathrm{i}y) \to \quad & \mathrm{e}^{\mathrm{i}\phi}(x + \mathrm{i}y) = (x' + \mathrm{i}y') \quad \to \\ & \left(\vartheta' = \frac{\pi}{4},\ \phi' = |\phi| \bmod \pi,\ \varphi' = \left|\sqrt{\xi_x \xi_y}\right| \bmod \pi\right)\end{aligned} \tag{9}$$

Note also that the triplet of products of all possible operators $\mathrm{e}^{\pm\psi_1}\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\pm\psi_2}$ belongs to the group $SL(2, \mathbb{R})$, namely

$$\begin{aligned}&\begin{pmatrix} \mathrm{e}^{\psi_1} & 0 \\ 0 & \mathrm{e}^{-\psi_1} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\psi_2} & 0 \\ 0 & \mathrm{e}^{-\psi_2} \end{pmatrix} = \\ &\begin{pmatrix} \mathrm{e}^{\psi_1+\psi_2}\cos\phi & \mathrm{e}^{\psi_1-\psi_2}\sin\phi \\ -\mathrm{e}^{\psi_2-\psi_1}\sin\phi & \mathrm{e}^{-\psi_1-\psi_2}\cos\phi \end{pmatrix} \in SL(2, \mathbb{R})\end{aligned} \tag{10}$$

where $\phi, \psi_1, \psi_2 \in \mathbb{R}$. Moreover, the group $SL(2, \mathbb{R})$ is generated by groups of special diagonal and orthogonal matrices $SD(2, \mathbb{R}), SO(2, \mathbb{R})$, since any simultaneously centroaffin and equiaffin transformation of the Cartesian plane can be obtained as a sequence of Euclidean rotations and equiaffin compressions (stretches) of the Cartesian plane coordinates. Thus, if we take an arbitrary element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the group $SL(2, \mathbb{R})$, then, equating it to a matrix representing the product of $\mathrm{e}^{\pm\psi_1}\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\pm\psi_2}$, we get

$$\tan^2\phi = \left|\frac{bc}{ad}\right|, \quad \psi_1 = \frac{1}{4}\log\left|\frac{ab}{cd}\right|, \quad \psi_2 = \frac{1}{4}\log\left|\frac{ac}{bd}\right| \tag{11}$$

whence follows

$$\psi_1 + \psi_2 = \frac{1}{2}\log\left|\frac{a}{d}\right|, \quad \psi_1 - \psi_2 = \frac{1}{2}\log\left|\frac{b}{c}\right| \tag{12}$$

and since $ad - bc = 1$, then

$$\tan^2\phi = \left|\frac{bc}{bc + 1}\right| \tag{13}$$

Thus, if you require the preservation of the inverse image of the point of reference of the torus, are wound on a sphere with punctured poles (i.e. preservation of the nodes of the integer lattice Cartesian plane), we get the modular group $PSL(2,\mathbb{Z})$, and dividing the group $PSL(2,\mathbb{Z})$ into equivalence classes under the relation of equality of values of angular parameter $\vartheta = \arctan e^{\psi_1 \pm \psi_2}$, we obtain such a partition $PSL(2,\mathbb{Z})$ on factor set $PSL(2,\mathbb{Z})/\vartheta$, that in the class of equivalence will remain only matrix with a divisors of numbers $n = bc$, $n+1 = ad$ and quotients of division

$$\begin{pmatrix} a \big| (n+1) & n/c \\ c \big| n & (n+1)/a \end{pmatrix} \tag{14}$$

In turn, if we are interested in the factor set $PSL(2,\mathbb{Z})/\phi$, then its equivalence classes can be enumerated by a set of indexes $\left\{\frac{n}{n+1}\right\}$, and therefore the equivalence class of this factor set contains matrices

$$\begin{pmatrix} 1 & b \\ c & bc+1 \end{pmatrix} \tag{15}$$

where $n = bc$, moreover, if $c = 1$, then b is a prime number and the equivalence class consists of a single matrix

$$\begin{pmatrix} 1 & p \\ 1 & p+1 \end{pmatrix} \tag{16}$$

where $p \in \mathbb{P}$. And since

$$(1+\mathrm{i}) \begin{pmatrix} 1 & p \\ 1 & p+1 \end{pmatrix} = (p+1) + (p+2)\mathrm{i} \tag{17}$$

that is, we can say that prime numbers generate closed torus windings wound once on the sphere. Note also that the vector $(n, n+1)$ has a pseudo-Euclidean length $\sqrt{n(n+1)}$, and the winding on the torus specified by this vector is closed, and if p is a prime number, then the corresponding winding $(p, p+1)$ does not intersect with any other winding $(n, n+1)$ of smaller length anywhere except at the zero point. Indeed, since the windings can only intersect on the set circle of the torus, on which each n - winding occurs at points $2\pi \left\{\frac{m}{n}\right\}$, where $m, n \in \mathbb{N}$, and $n < p$, then due to inequality

$$\left\{\frac{m}{n}\right\} \neq \left\{\frac{m'}{p}\right\} \tag{18}$$

p - the winding does not intersect the n - winding anywhere except at the zero point. Thus, there is a factorization of multiple points of intersection of the windings $(n, n+1)$ and the torus setting circle, as a result of which only simple windings $(p, p+1)$ remain out of all possible windings, delivering all possible single points of intersection of the torus setting circle.

3 Sphere winding representation for theta and zeta functions

Further, let the graph of the linear function $y = \tau x$ be given on the Cartesian plane, where $\tau = \tan\vartheta > 0$, and on the corresponding complex plane it is represented by the linear function $z = (1 + \mathrm{i}\tau)x$. Then the expression

$$\mathrm{e}^{\pi \mathrm{i} y x} = \mathrm{e}^{\pi \mathrm{i} \tau x^2} = \mathrm{e}^{\pi \mathrm{i} \tan\vartheta x^2} \tag{19}$$

should be considered as a complex function of winding a graph line on a Villarceau circle of unit radius. However, since $\psi = \frac{1}{2}\log\left|\frac{y}{x}\right|$, where $\left|\frac{y}{x}\right| = \tan\left(\frac{\vartheta}{2} + \frac{\pi}{4}\right)$, and the hyperbolic angle of inclination of a straight line passing through the center of the Cartesian plane and the point of the hyperbola $(x, y) = (\frac{1}{y}, y)$ is equal to

$$\psi = \frac{1}{2}\log y^2 \tag{20}$$

then

$$\begin{aligned} \phi &= \left|\mathrm{e}^{\psi}\right| \bmod 2\pi - \pi = |y| \bmod 2\pi - \pi \\ \vartheta &= 2\arctan\left(\mathrm{e}^{2\psi}\right) - \frac{\pi}{2} = \mathrm{gd}\, 2\psi \end{aligned} \tag{21}$$

where ϕ is the angular coordinate of the point $(\frac{1}{y}, y)$ on the torus, and $\vartheta = \phi/2 = \pi\{y\} - \frac{\pi}{2}$ is the corresponding latitude on the two-layer shell of the sphere.

Let us now try to combine the hyperbolic angle of the winding and the Euclidean angle of rotation of the winding in the mapping of a linear function $z = (r + \mathrm{i}\tau)x + a + \mathrm{i}b$, where $\tau > 0$, and $r \in \mathbb{R}$, to an exponent

$$\mathrm{e}^{\pi(r-1+\mathrm{i}\tau)x^2 + 2\pi c} = \mathrm{e}^{\pi(r-1)x^2 + 2\pi a}\, \mathrm{e}^{\pi \mathrm{i}\tau x^2 + 2\pi \mathrm{i} b} \tag{22}$$

where $c = a + ib$, which should be considered as a complex function of winding the graph line on a logarithmic spiral

$$e^{2\pi c} e^{\pi\left(\frac{r-1}{\tau}+i\right)t} \tag{23}$$

where $t = \tau x$. However, this winding can be revived if the variable x is given a time value, the τ parameter is the value of the angular velocity of the spiral unfolding, and the $\frac{r-1}{\tau}$ parameter is the value of the tangent of the logarithmic spiral. Finally, bearing in mind that $\psi = \frac{1}{2}\log \pi(r-1)$ is the hyperbolic angle of inclination of a straight line $y = \pi(r-1)x$ at time $x = 1$ and $\psi = \frac{1}{2}\log \pi(r-1)x^2$ at time x, we write out explicitly the mapping of the winding of a complex linear function to a complex spiral in the representation of the winding of a sphere

$$\vartheta = 2\arctan\left(e^{\pi(r-1)x^2+2\pi a}\right) - \frac{\pi}{2}, \quad \varphi = \left|\pi\left(\tau x^2 + \frac{1}{2}b\right)\right| \bmod 2\pi \tag{24}$$

which puts each complex linear function (as well as the sum and product of two complex linear functions) in a one-to-one correspondence with a spiral line on the sphere.

Now let the real value $\tau \geq 0$ play the role of an evolutionary parameter, $\log \tau$ serves as a linear time parameter, and the image of a point on the winding is described by the following sawtooth expression

$$\left/ e^{\pi i \tau^\alpha x^2} \right/ = e^{-\pi\tau x^2}\left(\left| e^{\pi(1-\gamma)\tau x^2} + \pi \right| \bmod 2\pi - \pi\right) e^{\pi i \beta \tau x^2} \tag{25}$$

where the complex parameter $\alpha = \beta + i\gamma$ acts as the angular velocity of a point on the sphere. Note that the expression 25 consists of several multipliers, among which the first factor is the weight coefficient of the point, and the rest specify the coordinates of the point on the shell of the sphere. Complementing the description of the continuous process of evolution of the winding with the line of the observer, we obtain the expression

$$\left/ e^{\pi i \tau^\alpha x^2 + 2\pi i c x} \right/ = e^{-\pi\tau x^2}\left(\left| e^{\pi(1-\gamma)\tau x^2 - 2\pi b x} + \pi \right| \bmod 2\pi - \pi\right) e^{\pi i \beta \tau x^2 + 2\pi i a x} \tag{26}$$

where $c = a + ib \in \mathbb{C}$ is the complex parameter of the observer's angular velocity. However, if you fix the location of the points of measurement (observation) coordinates on the sphere, setting $x = m \in \mathbb{Z}$, then you can observe the winding point with coordinates

$$\left/ e^{\pi i \alpha \tau m^2 + 2\pi i m c} \right/ \tag{27}$$

and the sum of

$$\theta(c, \tau^\alpha) = \sum_{m \in \mathbb{Z}} \left/ e^{\pi i \alpha \tau m^2 + 2\pi i m c} \right/ \tag{28}$$

is a Jacobi theta function. At the same time, if we fix the measurement point and the position of the observer, setting $x = 1$, $c = 0$, then at discrete moments of evolution $\tau = \log n$, where $p \in \mathbb{N}$, you can observe the coordinates of the winding

$$/e^{\pi i \alpha \log n}/ = \frac{1}{n^\pi} \left(\left| n^{\pi - \gamma} + \pi \right| \bmod 2\pi - \pi \right) n^{\pi i \beta} \tag{29}$$

and the sum of

$$\zeta(s) = \sum_{n \in \mathbb{N}} /n^{-s}/ \tag{30}$$

where $s = \sigma + it = -\pi i \alpha$, represents the Riemann zeta function. In this case, the trivial zeros of the Riemann zeta function are the solutions of the equation

$$\sum_{n \in \mathbb{N}} \frac{1}{n} \left(\left| n^{1-\sigma} + 1 \right| \bmod 2 - 1 \right) = 0 \tag{31}$$

the physical meaning of which is to achieve a uniform distribution of the weight of points lying on the countable set of Villarceau circles of the toroidal shell of the sphere, with a fixed angular coordinate of the point on the Villarceau circle itself. At the same time, when a point moves at a speed of t along the Villarceau circle, which at discrete time $\tau = \log n$ has the hyperbolic angle $\psi = \frac{1}{2} \log n^2$, that is, when the Villarceau circle moves at a hyperbolic speed $\sigma = \frac{1}{2}$, coinciding with the rate of increase in the latitude of the toroidal shell of the sphere, the equation

$$\sum_{n \in \mathbb{N}} \frac{1}{n} \left(\left| \sqrt{n} + 1 \right| \bmod 2 - 1 \right) n^{it} = 0 \tag{32}$$

delivers all nontrivial zeros of the Riemann zeta function.

4 On a random walk along the broken lines of the winding of a sphere

Let be a finite set of natural numbers $\{1,\ldots,N\}$, and the selection from this set is random. Let's denote X a random variable that takes the value of the selected number. The choice is made so that the probability of choosing n is $P(X=n)=\frac{1}{N}$, that is, an equal probability of choice is assumed. In addition, let the complex random variable $Y_{\eta,N}=X^{\mathrm{i}\eta}$ be associated with this random variable, where i is an imaginary unit, η is a real parameter, which should be understood as a random point $\mathrm{e}^{\mathrm{i}\alpha}$, where $\alpha=2\pi\{\eta\ln X\}$, on the complex unit circle. Moreover, the probability of choosing $n^{\mathrm{i}\eta}$ is also equal to $P(Y_{\eta,N}=n^{\mathrm{i}\eta})=\frac{1}{N}$, and the complex value $\overline{P}(Y_{\eta,N}=n^{\mathrm{i}\eta})=\frac{n^{\mathrm{i}\eta}}{N}$ we treat as *probability amplitude* of selecting a random variable $n^{\mathrm{i}\eta}$. Then the mathematical expectation of a random variable $Y_{\eta,N}$ is calculated using the formula

$$M(Y_{\eta,N})=\sum_{1}^{N}\frac{n^{\mathrm{i}\eta}}{N}=\sum_{1}^{N}\overline{P}(n^{\mathrm{i}\eta}) \tag{33}$$

and since $Y_{\eta,N}=\mathrm{e}^{\mathrm{i}\alpha}$, where $\alpha=2\pi\{\eta\log X\}$ — the corner parameter is the unit circle of the complex plane, then the mathematical expectation $M(Y_{\eta,N})$ can be interpreted as the average value of a set of numbers $\{1^{\mathrm{i}\eta},\ldots,N^{\mathrm{i}\eta}\}$, which lies on the unit circle of the complex plane. This raises the question of the existence of a limit for which (if it exists) the equation is performed

$$\Psi_{\eta}(\beta)=\lim_{N\to\infty}\sum_{\{n|\ \alpha<\beta\}}\frac{n^{\mathrm{i}\eta}}{N}=\int_{0}^{\beta}\rho_{\eta}(x)\mathrm{d}x \tag{34}$$

where $\alpha=2\pi\{\eta\log X\}$, $0\le\beta\le 2\pi$, and $\rho_{\eta}(x)$ is a continuous complex-valued function. In this case, the function $\Psi_{\eta}(\beta)$ should be interpreted as a function of the continuous distribution of the probability amplitude of the random variable Y_{η}, which is obtained from the discrete random variable $Y_{\eta,N}$ when N tends to infinity. In fact, since

$$\Psi'_{\eta}(\beta)=\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\sum_{\{n|\ \beta<\alpha<\beta+\varepsilon\}}\frac{n^{\mathrm{i}\eta}}{\varepsilon N}=\rho_{\eta}(\beta) \tag{35}$$

where due to the $n^{i\eta} = e^{i\beta}$ takes place

$$|\rho_\eta(\beta)| = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \sum_{\{n|\ \beta<\alpha<\beta+\varepsilon\}} \frac{1}{\varepsilon N} \tag{36}$$

then the function $|\rho_\eta(x)|$ is the probability density of the random variable Y_t, and the function $\rho_\eta(x) = e^{ix}|\rho_\eta(x)|$ — respectively, the density of the probability amplitude of the random variable Y_η. Finally, note that

$$M(Y_\eta) = \Psi_t(2\pi) = \lim_{N\to\infty} \sum_1^N \frac{n^{i\eta}}{N} = \int_0^{2\pi} \rho_\eta(x)\mathrm{d}x \tag{37}$$

So far, we have assumed an equal probability of choosing a random variable Y_η, however, if, for example, $P(Y_\eta) = \frac{n^{-\sigma}}{\zeta(\sigma)}$, where $\sigma > 1$, then

$$M(Y_\eta) = \frac{1}{\zeta(\sigma)} \sum_1^\infty n^{i\eta-\sigma} = \frac{\zeta(\sigma + i\eta)}{\zeta(\sigma)} \tag{38}$$

therefore it is reasonable to assume that

$$M(Y_\eta) = \frac{\zeta(\sigma + i\eta)}{\zeta(\sigma)} \tag{39}$$

and with $\sigma < 1$. It should be borne in mind that the relative probability of choosing a random variable Y_η (the probability taken without taking into account the normalization factor) is equal to $n^{-\sigma}$.

Next, based on the dynamics of Brownian motion [8], we consider the process of random walk of a particle along the winding of a sphere. Let the probability of free run without taking into account the normalizing multiplier for the time interval τ from the point $x = 0$, to the point $x = m$ on the winding, which rotates at the complex speed $\alpha = \beta + i\gamma$, where $\gamma > 0$, is equal to $e^{-\pi\gamma\tau m^2}$. Then, combining this probability with reliable information about the coordinates of the particle on the sphere winding in one complex number

$$e^{\pi i\tau^\alpha m^2} = e^{-\pi\gamma\tau m^2} e^{\pi i\beta\tau m^2} \tag{40}$$

we get the amplitude of the free run probability for the time interval τ from the point $x = 0$, to the point $x = m$ on the winding rotating at the complex speed $\alpha = \beta + \mathrm{i}\gamma$. Now let the measurement time be discrete $\tau = n$, and in the sum of the probability amplitudes

$$\sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} \tag{41}$$

contains information about the average (time) value of the particle coordinate on the sphere winding (the sum argument) and the intensity of this average value (the sum module). Strictly speaking, in this case, the sum of the probability amplitudes is the mathematical expectation of such a random variable as a complex representation of the angular coordinate of the sphere winding. In turn, the probability amplitude of the Markov random walk process consisting of a countable number of arbitrary free runs on not yet occupied windings of the sphere is the product of sums

$$\prod_{n\in\mathbb{P}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} \tag{42}$$

that is, the infinite product of discrete values of the theta function. I do not undertake directly to calculate the function to which this infinite product of infinite sums converges, but we will look at it from a different (more physical) side. Let's assume for simplicity that $n^{\alpha} = n^{\beta}$, and therefore all random events (arbitrary free runs of the particle on an arbitrary winding of the sphere) have an equal probability. In addition, let be given a finite time measurement interval and the coordinates of the particle $\Delta\tau = \Delta x$, and $\tau_n = n\Delta\tau$, where $n \in \mathbb{N}$, $x_j = j\Delta x$, where $j \in \mathbb{Z}$, and $f\left(\tau_1, x_j(\tau_1)\right)$ is the initial value of the complex amplitude of the probability of occurrence of a particle with the coordinate x_j. Then the product of the sums

$$f\left(\tau_n, x_m(\tau_n)\right) = \prod_{i<n} \sum_{j,k\in\mathbb{Z}} f\left(\tau_1, x_j(\tau_1)\right) \mathrm{e}^{\pi \mathrm{i} \tau_i^{\beta} \left(\frac{x_k(\tau_{i+1}) - x_j(\tau_i)}{\Delta\tau}\right)^2} \tag{43}$$

for extremely small time measurement intervals and the coordinates of the particle $\Delta\tau = \Delta x \to 0$ can be denoted by a generalizing functional integral

$$f(\tau \to \infty, x) = \int Dx f(\tau, x) \mathrm{e}^{\pi \mathrm{i} d \tau^{\beta} \left(\frac{dx}{d\tau}\right)^2} \tag{44}$$

from which the integral equation follows

$$f(\tau + \Delta\tau, x) = \int\limits_{-\infty}^{+\infty} f(\tau, y) \mathrm{e}^{\pi \mathrm{i} \Delta \tau^{\beta} \left(\frac{x-y}{\Delta\tau}\right)^2} \mathrm{d}y = \int\limits_{-\infty}^{+\infty} f(\tau, y) \mathrm{e}^{\pi \mathrm{i} \tau^{\beta-1} \frac{(x-y)^2}{\Delta\tau}} \mathrm{d}y \tag{45}$$

and the corresponding differential equation

$$\mathrm{i} \frac{\partial f(\tau, x)}{\partial \tau} = -\frac{\tau^{1-\beta}}{4\pi} \frac{\partial^2 f(\tau, x)}{\partial x^2} \tag{46}$$

and then compare it with the one-dimensional Schrödinger equation of a free particle. Note that when replacing $t = \log \tau$, corresponding to the transition to linear time, the equation takes the form

$$\mathrm{i} \frac{\partial \psi(t, x)}{\partial t} = -\frac{\mathrm{e}^{-t\beta}}{4\pi} \frac{\partial^2 \psi(t, x)}{\partial x^2} \tag{47}$$

This equation, of course, differs from the Schrödinger equation, but in the special case $\beta = 0$ the difference disappears. However, it is clear that an even greater generalization of the Schrödinger equation is possible

$$\mathrm{i} \frac{\partial \psi(t, x)}{\partial t} = -\frac{\mathrm{e}^{-t\alpha}}{4\pi} \frac{\partial^2 \psi(t, x)}{\partial x^2} \tag{48}$$

where $\alpha = \beta + \mathrm{i}\gamma$.

In conclusion, let's take the statistical sum

$$\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} \tag{49}$$

which is essentially the average value of the angular coordinates of the sphere's windings, obtained by summing not only the set of all integer values of the observer's line coordinates, but also the natural set of all possible graphs of the sphere's winding, and establishing its relationship with the

Jacobi theta function $\vartheta(0,\tau)$. First of all, note that the sum 49 contains two sums, namely, the Jacobi theta function

$$\sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} \alpha n m^2} = \theta(0, n\alpha) \tag{50}$$

where $\alpha = \beta + \mathrm{i}\gamma$ and the sum

$$\sum_{n\in\mathbb{N}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} = \sum_{n\in\mathbb{N}} \mathrm{e}^{\pi \mathrm{i} \alpha n m^2} \tag{51}$$

which in case $\gamma > 0$ is a convergent geometric progression and is equal to

$$\frac{\mathrm{e}^{\pi \mathrm{i} \alpha m^2}}{1 - \mathrm{e}^{\pi \mathrm{i} \alpha m^2}} \tag{52}$$

Therefore, the original double sum 49 can be simplified to a simple sum and get the desired function

$$\sum_{m\in\mathbb{Z}} \frac{\mathrm{e}^{\pi \mathrm{i} \alpha m^2}}{1 - \mathrm{e}^{\pi \mathrm{i} \alpha m^2}} = \frac{\theta(0,\alpha)}{1 - \theta(0,\alpha)} \tag{53}$$

Thus, for a statistical sum interpreted as the probability amplitude of a gas consisting of particles wandering around the winding of a sphere, the identity is fulfilled

$$\sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^{\alpha} m^2} = \sum_{n\in\mathbb{N}} \theta(0, n\alpha) = \frac{\theta(0,\alpha)}{1 - \theta(0,\alpha)} \tag{54}$$

However, if the function

$$\rho(\tau) = \frac{\theta(0, \mathrm{i}\tau)}{1 - \theta(0, \mathrm{i}\tau)} \tag{55}$$

where $\tau \geq 0$, is understood as the complex probability density, then applying the integral transformation to it, we get the frequency density of the probability distribution (the characteristic function)

$$\zeta(z) = \int_0^{+\infty} \frac{\theta(0, \mathrm{i}\tau)\tau^{\mathrm{i}z-1}}{1 - \theta(0, \mathrm{i}\tau)} \mathrm{d}\tau = \int_{-\infty}^{+\infty} \frac{\theta(0, \mathrm{i}\mathrm{e}^{t})\mathrm{e}^{\mathrm{i}zt}}{1 - \theta(0, \mathrm{i}\mathrm{e}^{t})} \mathrm{d}t \tag{56}$$

where $t = \log \tau$. And since the evolution of the function $\theta(x, \mathrm{i}\tau)$is described by the heat equation

$$\frac{\partial \theta(x, \mathrm{i}\tau)}{\partial \tau} = \frac{1}{4\pi} \frac{\partial^2 \theta(x, \mathrm{i}\tau)}{\partial x^2} \tag{57}$$

with the initial conditions $\lim_{\tau \to 0} \theta(x, \mathrm{i}\tau) = \delta(x - m)$, where $m \in \mathbb{Z}$,it would be interesting to know which differential equation corresponds to the function

$$\varphi(x, \mathrm{i}\tau) = \sum_{n \in \mathbb{N}} \theta(x, n\mathrm{i}\tau) = \frac{\theta(x, \mathrm{i}\tau)}{1 - \theta(x, \mathrm{i}\tau)} \tag{58}$$

or function

$$\zeta(x, \mathrm{i}z) = \int_0^{+\infty} \tau^{\mathrm{i}z-1} \varphi(x, \mathrm{i}\tau) \mathrm{d}\tau \tag{59}$$

obtained from the original one using the Mellin integral transform.

5 Conclusion

Thus, despite the rejection of the physicality of the dynamic model presented here, its abstract mathematical constructions have some similarities with the mathematical formalism of quantum mechanics, and therefore the questions of the substantiation of quantum mechanics could possibly be answered within the framework of the mathematical formalism developed here. At least, if the function 58 satisfies the generalized Schrödinger equation, then it could be interpreted as the amplitude of oscillations of the sphere winding. In addition, the unitary group of isometries of a torus on a sphere tells us that the nature of the electron is somehow related to the geometry of the sphere's winding. For example, the fine-structure constant could be associated with the angle of inclination of the Villarceau circle, which it has at the present moment of evolution, and the electron mass - with the pseudo-Euclidean length of the closed winding of the sphere, which has the shape of a (2,3)-toric knot. Then the hierarchy of leptons could be explained by a bunch of trefoils. Indeed, let the tangent of the angle of inclination of

the prototype of a closed winding on the complex plane be equal to

$$\tan \vartheta_c = \frac{2}{3}\alpha \tag{60}$$

where $\tan \vartheta_0 = \alpha$ is the tangent of the angle of inclination of the diagonal of the rectangle, which serves as a prototype of the torus winding on the sphere, and the growth coefficient of the derivative of this winding of the logarithmic spiral $r = \mathrm{e}^{k\varphi}$ is equal to

$$k = -1 - \log \frac{2}{3}\alpha \tag{61}$$

Then, if the unit of measurement of the sweep angle φ of the spiral is taken to be the number of quarter turns of the winding, and as the lepton mass of the N-th hierarchy, we take the length of the spiral for $j + 1$ of the quarter turns of the winding

$$M_N = \mathrm{e}^{k(1+j)} \tag{62}$$

where $j = \sum_{n=0}^{N} n^4$, then with the simplification $k' = k + 1$, the Barut formula [9] for the lepton mass ratio is fulfilled.

Declarations

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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HYPERSTRUCTURES ASSOCIATED WITH CHARACTER ALGEBRAS AND COLOR SCHEMES[1]

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Abstract

It is the purpose of this paper to provide examples of hyperstructures associated with the study of symmetry in physics and chemistry. In applications the symmetries of an object are typically considered as a group; so most of the examples presented are connected to constructions from group theory. In particular, we show that a natural hypergroup is associated with every character algebra. This unifies several classical hypergroup constructions. We also show how certain edge colorings of graphs give raise to hypergroups with special properties.

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1. Introduction.

The study of symmetry is closely related to the theory of groups and its extensions. Symmetry groups have been widely applied in chemistry [12], crystallography [17], and solid-state physics [1]. In these areas distinctions are usually made between *symmetric objects* (planes, axes, etc) and *symmetric operations* (reflections, rotations, etc) which leave an object invariant. In this paper *symmetry* will refer to symmetric operations. Many applications of groups involving symmetries have involved coset decompositions, double coset decompositions, decompositions into conjugacy classes, and group characters. Below we introduce a hypergroup structure on each of these "spaces". This paper is intended to be a source of ideas about constructions to show a non-specialist that hypergroups are fairly naturally associated with objects with which they have familiarity.

Section 3 contains a specific result. Namely, we show that a C-algebra in the sense of Y. Kawada [18] gives raise to a special hypergroup called a quasi-cannonical hypergroup by Bonansinga and Corsini [4] and a polygroup in [6].

A *polygroup* is a system $\langle M, \bullet, e\rangle$ where $e \in M$, $\bullet$ assigns a nonempty subset of M to each pair of elements in M, and the following axioms hold for all $x, y, z \in M$:

(1) for each x there exists a unique $x^{-1} \in M$ such that $e \in x \bullet x^{-1}$ and $e \in x^{-1} \bullet x$,

(2) $e \bullet x = \{x\} = x \bullet e$,

(3) $x \in y \bullet z$ implies $y \in x \bullet z^{-1}$ and $z \in y^{-1} \bullet x$,

(4) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$.

Sections 4 and 5 deal with connections between hypergroups and colorings of designs or graphs. In Section 4 we describe the principal coloring associated with a subgroup of a group and in Section 5 we look at polygroups derived from an edge coloring of a complete graph that satisfies a regularity condition.

Throughout, we illustrate constructions using the dihedral group D_4. This group is generated by a counter-clockwise rotation r of 90° and a horizontal reflection h. The group consists of the following 8 symmetries:

$$\{1 = r^0, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}.$$

Figure 1 below shows the actions of r, h, v, d, and f.

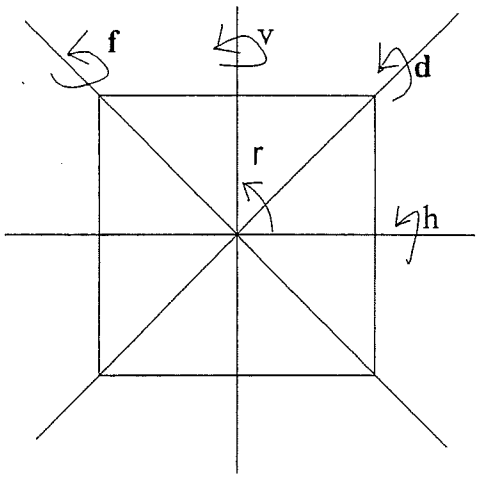


Figure 1 : D_4

2. Classical hypergroup constructions.

Coset Hypergroups. For a group G and a subgroup H let G/H denote the collection of all right cosets $Hg = \{hg: h \in H\}$ for g in G. The system $\langle G/H, \bullet, H\rangle$ is a hypergroup where

$$(Hg_1) \bullet (Hg_2) = \{Hg : g \in g_1 Hg_2\}.$$

The system G/H is called a *D-hypergroup* and is the primary example of a cogroup. These systems have been studied by Eaton [15], Krasner [19], and Utumi [25]. Recently Y. Sureau [24] has characterized all cogroups by nested triples of permutation groups.

As an example of the construction consider the subgroup $H = \{1, h\}$ of the group D_4. The right coset decomposition of D_4 with respect to H is

$$\{1, h\} \cup \{r, d\} \cup \{s, v\} \cup \{t, f\} .$$

If two cosets are multiplied elementwise the result may not be a single coset. For example, consider the product $(Ht)(Hr)$ in D_4. This part of the group table for D_4 is displayed below.

D_4	·	r	d	·
·	·	·	·	·
t	·	1	v	·
f	·	h	s	·

We see that $(Ht)(Hr) = H \cup Hs$. Thus, in the operation table for D_4/H the product $Ht \cdot Hr$ is recorded as the two elements H, Hs. Using 1, r, s, and t as coset representatives, the complete operation table for the cogroup D_4/H becomes

	H	Hr	Hs	Ht
H	H	Hr	Hs	Ht
Hr	Hr, Ht	Hs, H	Ht, Hr	H, Hs
Hs	Hs	Ht	H	Hr
Ht	Ht, Hr	H, Hs	Hr, Ht	Ht, H

In Section 4 we show how cogroups such as the one above derived from coset decompositions represent an algebra of colors associated with the subgroup.

Double coset algebras. The unit element of a cogroup is normally a one-sided scalar, not a 2-sided scalar. This is overcome by looking at double cosets. A

double coset HgH of a subgroup H of a group G is the collection of all distinct elements h_1gh_2 where h_1 and h_2 range over H. The double cosets of H give a decomposition of G. Let $G//H$ denote the collection of all double cosets of G with respect to H. A *double coset algebra* is a hypergroup $\langle G//H, \bullet, H\rangle$ where

$$(Hg_1H)\bullet(Hg_2H) = \{Hg_1hg_2H : h \in H\}.$$

The system $G//H$ is a example of a polygroup or quasi-cannonical hypergroup.

If $H = \{1, h\}$ is the subgroup of D_4 considered earlier, the double coset decomposition is

$$\{1, h\} \cup \{r, d, t, f\} \cup \{s, v\} .$$

As with the elementwise product of two cosets the elementwise product of double cosets can also produce more than one double coset. Consider the product $(HrH)(HrH)$. The relevant part of the group table of D_4 is displayed below.

D_4	·	r	d	t	f	·
·	·	·	·	·	·	·
r	·	s	h	1	v	·
d	·	v	1	h	s	·
t	·	1	v	s	h	·
f	·	h	s	v	1	·
·	·	·	·	·	·	·

We see that $(HrH)(HrH) = H \cup HsH$; thus the polygroup product $HrH \bullet HrH$ is the two elements H, HsH. The full table for the polygroup $D_4//H$ is given below. Notice that the unit element H is a scalar element.

	H	HrH	HsH
H	H	HrH	HsH
HrH	HrH	H, HsH	HrH
HsH	HsH	HrH	H

Conjugacy class polygroups. In dealing with a symmetry group two symmetric operations belong to the same class if they represent the same map with respect to (possibly) different coordinate systems where one coordinate system is converted into the other by a member of the group (cf., [12]). In the language of group theory this means elements a, b in a symmetry group G belong to the same class if there exist a $g \in G$ such that $a = gbg^{-1}$, i.e., a and b are conjugate. The collection of all conjugacy classes of a group G is denoted by $\bar{G}$ and the system $\langle \bar{G}\bullet, \{e\} \rangle$ is a polygroup where e is the identity of G and the product $A \bullet B$ of conjugacy classes A and B consists of all conjugacy classes contained in the elementwise product AB. This hypergroup was recognized by Campaigne ([5]) and by Dietzman ([14]).

In the case of D_4 there are 5 conjugacy classes: $\{1\}$, $\{s\}$, $\{r,t\}$, $\{d,f\}$, and $\{h,v\}$. Let us denote these classes as $C_1,\ldots, C_5$ respectively. Then the polygroup $\bar{D}_4$ is

	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	C_1,C_2	C_5	C_4
C_4	C_4	C_4	C_5	C_1,C_2	C_3
C_5	C_5	C_5	C_4	C_3	C_1,C_2

As a sample of how to calculate the table entries consider $C_3 \cdot C_3$. To determine this product compute the elementwise product of the conjugacy classes $\{r,t\}\{r,t\} = \{s,1\} = C_1 \cup C_2$. Thus, $C_3 \cdot C_3$ consists of the two conjugacy classes C_1, C_2.

Character polygroups. Closely related to the conjugacy classes of a finite group are its characters. Let $\hat{G} = \{\chi_1, \chi_2, \cdots, \chi_k\}$ be the collection of irreducible characters of a finite group G where χ_1 is the trivial character. The character polygroup $\hat{G}$ of G is the system $\langle \hat{G}, \bullet, \chi_1 \rangle$ where the product $\chi_i \bullet \chi_j$ is the set of irreducible components in the elementwise product $\chi_i \chi_j$. The system $\hat{G}$ was investigated by R. Roth [21] who considered a duality between $\hat{G}$ and $\overline{G}$.

Before calculating $\hat{D}_4$ we need to know the 5 irreducible characters of the dihedral group D_4. These are given by the following character table.. (Since characters are constant on conjugacy classes it is usual to list only the conjugacy classes across the top of the table.)

	C_1	C_2	C_3	C_4	C_5
χ_1 :	1	1	1	1	1
χ_2 :	1	1	-1	1	-1
χ_3 :	1	1	-1	-1	1
χ_4 :	1	1	1	-1	-1
χ_5 :	2	-2	0	0	0

We illustrate the calculation of the polygroup product of two characters by considering $\chi_5 \bullet \chi_5$. The pointwise product of χ_5 with itself yields the following (non-irreducible) character:

	C_1	C_2	C_3	C_4	C_5
$\chi_5\chi_5$:	4	4	0	0	0

This character can be written as a sum of irreducible characters in exactly one way: $\chi_5\chi_5 = \chi_1 + \chi_2 + \chi_3 + \chi_4$. This is indicated by the entry in the lower right

hand corner of the polygroup table for $\hat{D}_4$. In general the polygroup product of two characters $\chi_i \bullet \chi_j$ tells which irreducible characters are in the product $\chi_i\chi_j$ but not the multiplicity. Using i in place of the character χ_i the polygroup $\hat{D}_4$ is

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	1,2,3,4

3. A construction from C-algebras.

The above constructions of hypergroups based on double cosets, conjugacy classes, and characters all make use of the fact that when two objects of a type under consideration (i.e., double cosets, characters, etc) are multiplied in the natural way, the product is uniquely composed of objects in the set (perhaps with repetition). The polygroup binary operation is a type of convolution which tells which objects are in the product but not the multiplicity. We formalize this idea by showing that every C-algebra gives raise to a polygroup. The notion of C-algebra (or character algebra) presented here is due to Y. Kawada [18] (see also [2]) except that the commutativity requirement has been weakened.

A *C-algebra* is a classical algebra A over the complex numbers together with a basis $X = \{x_0, \cdots, x_d\}$ for A (as a complex linear space) such that

(C_1) A is an algebra and $x_i \bullet x_j = \sum_k p_{ij}^k x_k$ for all i, j,

(C_2) A has an identity element $e = x_0$, i.e., $p_{0j}^k = \delta_{jk} = p_{j0}^k$,

(C_3) every p_{ij}^k is a real number,

(C_4) there exist a permutation $i \neg i'$ (for $i = 0, \ldots, d$) such that $(i')' = i$ and

$p_{ij}^{k} = p_{j'i'}^{k'}$,

(C_5) $p_{ji}^{0} = p_{ij}^{0} = \kappa_i \delta_{ij'}$ with $\kappa_i > 0$ for all i, j, and

(C_6) the map $x_i \mapsto \kappa_i$ induces a linear representation of A.

The condition (C_4) implies that the map $x_i \mapsto x_{i'}$ extends to an anti-automorphism of A. A C-algebra is *commutative* if $p_{ij}^{k} = p_{ji}^{k}$ for all i, j, k. The lemma below summarizes a few elementary facts.

Lemma. (1) $0' = 0$,

(2) $\kappa_0 = 1$,

(3) $\kappa_i = \kappa_{i'}$,

(4) $\kappa_k p_{ij}^{k} = \kappa_i p_{kj'}^{i} = \kappa_j p_{i'k}^{j}$.

Proposition. *Every C-algebra A with basis X such that the parameters p_{ij}^{k} are all non-negative (the Kreim condition) determines a polygroup $Pg(A) = \langle X, \bullet, e \rangle$ where $x_i \bullet x_j = \{x_k : p_{ij}^{k} \neq 0\}$ and $x_i^{-1} = x_{i'}$ for all i, j.*

Proof. Since $x_i \bullet x_{i'} = \sum_k p_{ii'}^{k} x_k = \kappa_i x_0 + \cdots$ and $\kappa_i > 0$, it is clear that $x_0 \in x_i \bullet x_{i'}$. If $x_0 \in x_i \bullet x_{j'}$, $p_{ij}^{0} \neq 0$ which implies $j = i'$ by (C_5). Similarly, $x_{i'}$ is the only y such that $x_0 \in y \bullet x_i$, so axiom (1) holds. Axiom (2) follows from the C-algebra property (C_2) and axiom (3) from Lemma (4). For (4) notice that $x_u \in (x_i \bullet x_j) \bullet x_k \Leftrightarrow p_{ij}^{v} p_{vk}^{u} \neq 0$ if and only if $p_{ij}^{v} p_{vk}^{u} \neq 0$ for some v and similarly, $x_u \in x_i \bullet (x_j \bullet x_k)$ if and only if $p_{iv}^{u} p_{jk}^{v} \neq 0$ for some v. The associative law for $Pg(A)$ follows from the equality $\sum_v p_{ij}^{v} p_{vk}^{u} = \sum_v p_{iv}^{u} p_{jk}^{uv}$ (a consequence of (C_1)) and the Krein condition. □

Examples of C-algebras not only include the situations mentioned earlier, but also the adjacency algebras of association schemes ([2]), S-algebras over finite groups ([3]), and centralizer algebras of homogeneous coherent

configurations ([16]). The centralizer algebra of a coherent configuration is called a cellular algebra in the work of Weisfeiler ([27]) on the graph isomorphism problem.

4. Color symmetries

In Section 2 we showed how a coset decomposition gave raise to a cogroup. Coset decompositions are related to the study of color symmetries of designs and crystals. The original influence for this was the work of Shubnikov [23] on crystallography. The ideas illustrated here follow the treatment given by Roth [22]. Below we illustrate how a transitive symmetric coloring is associated with a coset decomposition. Since every transitive symmetric coloring is equivalent to one constructed this way, by the construction in Section 2, every transitive symmetric coloring has an associated cogroup. We first define color symmetries and symmetric colorings, then give their construction.

Suppose G is the group of symmetries of a design that has its regions colored (with finitely many colors). An element $g \in G$ is a *color symmetry* if it induces a permutation of the set of colors, i.e., whenever g maps a region colored i onto one colored j, then g must map all regions colored i onto regions colored j. The color symmetries form a subgroup of the group G of all symmetries of a design. If every element of G is a color symmetry, the assignment of colors to the regions of the design is called a *symmetric coloring*. We are interested in *transitive* symmetric colorings, i.e., symmetric colorings such that any color can be mapped into any other by a suitable element of G.

In order to set up a correspondence between the regions of a design and the elements of the symmetric group G of the design a *sequence of fundamental regions* of the design is selected. This is a collection of disjoint regions such that for any two regions in the collection there is a unique symmetry in G that maps one onto the other. Fixing such a sequence of regions allows a unique element of the symmetry group G to be assigned to each region. To start the assignment

select an arbitrary region Ω to correspond to the identity element of G. Then, for each fundamental region R, there is a unique $g \in G$ which maps Ω onto R. Assign the group element label "g" to the region R.

The "cross" in Figure 2 illustrates the labeling. This design has 8 fundamental regions and the dihedral group D_4 as its symmetry group. Selecting the region labeled Ω as the initial region, the other regions correspond to the group elements indicated.

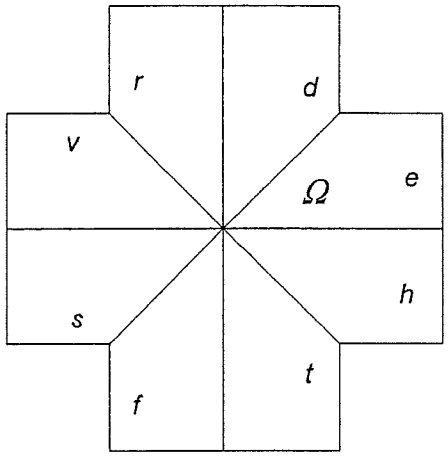


Figure 2: Region labels

If a design has a transitive symmetric coloring and H is the stablizer subgroup of the color of the Ω region, there is a one-one correspondence between the right cosets of H and the set of colors. This leads us to the *principal coloring* of the design associated with a subgroup H of finite index in G. Choose a set of colors in one-one correspondence with the elements of the cogroup G/H; say, use the colors 1,...,n where $G = H \cup Hx_2 \cup \ldots \cup Hx_n$ (with $x_1 = e$). Assign the color i to all regions labeled with elements in the coset Hx_i. This gives a transitive symmetric coloring of the design.

As an example, consider the subgroup $H = \{1,h\}$ of the dihedral group D_4. Assign $\{1,h\}$ ≒ white, $\{r,d\}$ ≒ red, $\{s,v\}$ ≒ blue, and $\{t,f\}$ ≒ green. The symmetric coloring of the design from H is given in Figure 3 below.

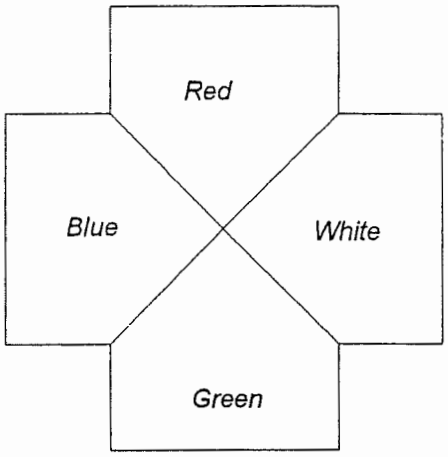


Figure 3: Coloring from H

Roth ([22]) shows that every transitive symmetric coloring (with the same sequence of fundamental regions) is equivalent to one obtained by the process above. This and much more is treated in [22]. The discussion above shows that a cogroup can be assigned to every transitive symmetric coloring of a design, namely, G/H if the coloring corresponds to the subgroup H of G.

	W	R	B	G
W	W	R	B	G
R	R,G	B, W	G, R	W, B
B	B	G	W	R
G	G, R	W, B	R, G	G, W

Using the obvious abbreviations W, R, B, and G for the colors white, red, blue,

and green respectively, the table above is the cogroup G/H assigned to the coloring in Figure 3. It is the same cogroup given in Section 2:

5. Color algebras

An *association scheme* [2] is a partition (or coloring) of the edges of a complete graph V in a very regular way. As mentioned in Section 3 the adjacency algebra of the collection of relations of an association scheme is a C-algebra. The notion of a color scheme ([6], [7], [9]) given below is a generalization of the idea of an association scheme and will have a natural polygroup, called a color algebra, associated with it. In the definition below "|" denotes relation composition.

Suppose C is a set (of colors) with a distinguished element $0 \in C$. (Think of 0 as a 'neutral' color.) Suppose $i : C \to C$ with i^2 equal the identity on C and $i(0) = 0$. (Think of $i(c)$ as the color 'opposite' to color c.) A *C-color scheme* is a system $\langle V, \{C_a : a \in C\}\rangle$ such that $C_a \subseteq V \times V$ for all $a \in C$ and

(1) $\{C_a : a \in C\}$ is a partition of $V \times V$ and C_0 is the identity on V,

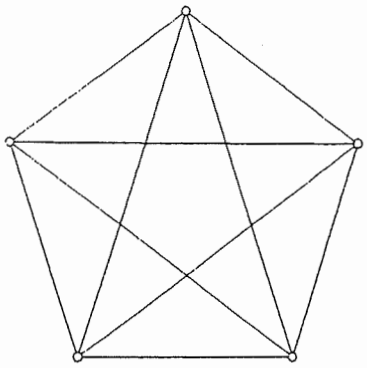
(2) $C_{i(a)} = C_a^{\smile}$ (the converse of C_a) for each $a \in C$,

(3) for every $a, b, c \in C$, $C_c \cap (C_a | C_b) \neq \varnothing$ implies $C_c \subseteq C_a | C_b$.

Condition (1) means that colors are assigned to all edges of a complete directed graph and Condition (2) means that for every edge the color assigned to the edge in the reverse direction is uniquely determined by the color on the original edge. Condition (3) means that whenever some c-colored edge is part of a triangle whose other sides are colored a and b, then every c-colored edge is part of such a triangle. Association schemes and homogeneous coherent configurations are examples of color schemes that satisfy the stronger property (3'), given below, which says that if one c-colored edge is part of k triangles whose other edges are colored a and b, then every c-colored edge is a part of k such triangles.

(3') for every a, b, c $\in C$ and x, y $\in V$, the number of elements in the set $\{z \in V : (x, z) \in C_a \text{ and } (z,y) \in C_b\}$ is independent of the edge (x,y) $\in C_c$.

The following 5 point example has two colors (solid and dash) in addition to the neutral color.



Color 0 = Neutral (points)

Color 1 = Solid line

Color 2 = Dashed line

Figure 4

The *color algebra* of a C-color scheme $\langle V, \{C_a : a \in C\}\rangle$ is the system $\langle C, *, 0\rangle$ based on the set of colors C with * defined for a, b $\in C$ by

$$a*b = \{c \in C : C_c \subseteq C_a | C_b\}.$$

It is easy to see that a color algebra is a polygroup and that the inverse of a color c is the color i(c). As an example of how the product operation works we form the color algebra of the undirected graph in Figure 4 above.

	0	1	2
0	0	1	2
1	1	0,2	1,2
2	2	1,2	0,1

Let 0 denote the neutral color, 1 the solid line color, and 2 the dotted line color. To compute 1*1 look at the composition $C_1|C_1$. It is $C_0 \cup C_2$. Likewise, we see that $C_1|C_2 = C_1 \cup C_2$ and $C_2|C_2 = C_0 \cup C_1$. With these calculations we see that the color algebra of the graph in Figure 4 has the operation table above:

For the record there are exactly 10 three element polygroups ([20], [9]).

A polygroup is called *chromatic* if it is isomorphic to the color algebra of some color scheme. Chromatic polygroups have the nice property that they are exactly the polygroups which have a faithful representation as a regular polygroup of generalized permutations. In other words, a strong form of the Cayley representation holds for these hypergroups. (See, Comer [9].) For another representation of hypergroups by generalized permutations, see [26].

We conclude with a few remarks about other applications of polygroups. Polygroups are closely related to the study of the theory of relations. In [7] an alternate to the relational calculus is proposed which is based on an extension of the notion of polygroup. The polygroupoid notion introduced there is a partial polygroup-like system with a set of identity elements instead of a single identity. To define these systems it is convenient to make the inverse operation explicit. In more detail, a *polygroupoid* is a partial hyperalgebraic structure $\langle A, \circ, I, ^{-1} \rangle$ where $\circ$ is a partial binary hyperoperation on A, $I \subseteq A$, and $^{-1}$ is a unary operation on A such that the following axioms hold for all $x, y, z \in A$:

(i) $(x \circ y) \circ z = x \circ (y \circ z)$

(ii) $x \circ I = x = I \circ x$

(iii) the formulas $x \in y \circ z$, $y \in x \circ z^{-1}$, and $z \in y^{-1} \circ x$ are equivalent.

Condition (i) should be interpreted as saying that if either side is non-empty, then both sides are non-empty and the sets are equal.

Polygroupoids correspond to the atom structures of systems of relations. In a sense polygroupoids (and the special case, polygroups) are a way to

combinatorially classify types of relations. One effort in this direction is presented in [8]. Another example deals with the association schemes associated with regular trees by Delsarte ([13]). In [10] the color algebras associated with regular trees are completely described. In [11] polygroupoids are generalized further to partial multi-valued loops. These systems arise in the representation of atom structures of 3-dimensional cylindric algebras.

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CONSTRUCTIONS OF HYPERGROUPS

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Abstract

Different methods of constructing hypergroups starting from groups, a family of hypergroups, sets, graphs and other mathematical systems are presented in the paper. In this way, one may obtain a lot of examples of hypergroups, but one may also see some algebraic constructions in the class of hypergroups. The paper is a kind of survey, but some constructions are really new.

The references are strictly limited at papers presenting such constructions, in spite of the great interest of researches in other topics of the hypergroup theory.

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1 Historical remarks

The history of hypergroups started with the paper written by F. Marty [17] in 1934. The idea of considering such an algebraic structure came from the study of the set of all left (or right) cosets of a group G with respect to a subgroup S of it, $H = \{xS \mid x \in G\}$. If S is a normal subgroup, then H can be "naturally" endowed with a group operation:

$$xS.yS = xyS, \tag{1}$$

induced by the group operation on representatives. If S is not normal, then the right side of (1) depends upon the chosen representatives in the two involved classes and such an operation cannot be defined.

Therefore, one may consider a hyperoperation on H, i.e. a mapping

$$H^2 \to \mathcal{P}(\mathcal{H})^*(:= \mathcal{P}(\mathcal{H}) \setminus \{\emptyset\}),$$

defined by:

$$xS.yS = \{zS \mid z \in xSy\}.$$

It is easily seen that, for all xS, yS, zS in H, this hyperoperation is *associative*, that is:

$$(xS \cdot yS)\, zS = xS.(yS \cdot zS); \tag{2}$$

here we denote, for nonempty subsets A,B of H, $A \cdot B = \cup\{a \cdot b \mid a \in A, b \in B\}$.

Moreover, eS has the property:

$$xS.eS = xS \text{ and } xS \in eS \cdot xS, \tag{3}$$

$$eS \in (xS \cdot x^{-1}S) \cap (x^{-1}S \cdot xS) \tag{4}$$

$$xS \cdot H = H \cdot xS = H. \tag{5}$$

In the last twenty years, the study of hypergroups has got a new interest, because of the applications of algebraic hyperstructures in other fields of mathematics like geometry and combinatorics. There are already some books on hypergroups:

- W. Prenowitz, J. Jantosciak, *Join Geometries,* U.T.M. Springer, Berlin, New York, Heidelberg, 1979.
- P. Corsini, *Prolegomena of hypergroup theory,* Aviani Editore, Udine, 1993.
- T. Vougiouklis, *Hyperstructures and their representations,* Hadronic Press, 1994.

We also recall some of the first authors and papers on hypergroups, introducing classes of hypergroups and studying them:

- M. Dresher and O. Ore, *Theory of multigroups,* Amer. J. Math., **60** (1939), 705-733.
- J.E. Eaton, *Theory of cogroups,* Duke Math. J., **6** (1940), 101- 107.
- M. Krasner, *La caractérisation des hypergroupes de classes et le problème de Schreier dans les hypergroupes,* I, II, III, C.R. Acad. Sci. Paris, **212** (1941), 948-950; **218** (1944), 483-484, 542-544.
- H.S. Wall, *Hypergroups,* Amer. Math. J.,**59** (1937), 77-98.
- Y. Utumi, *On hypergroups of group right cosets,* Osaka Math. J., **1** (1949), 73-80.

2 Preliminaries

We recall first some definitions.

Definition 2.1 *A* **hypergroup** $\langle H, \cdot \rangle$ *is a nonempty set* H *endowed with a binary hyperoperation* $\cdot : H^2 \to \mathcal{P}(\mathcal{H})^*$, *such that the following axioms are satisfied:*

$$\forall x, y, z \in H,\ (x \cdot y) \cdot z = x \cdot (y \cdot z),\ \textit{(associativity law);} \tag{6}$$

$$\forall x \in H, x \cdot H = H \cdot x = H\ \textit{(reproduction law).} \tag{7}$$

We note that for the nonempty subsets A and B of H, we have:

$$A \cdot B := \cup\{a \cdot b \mid a \in A, b \in B\}.$$

Definition 2.2 *Two hypergroups* $\langle H, \cdot \rangle, \langle H', \cdot \rangle$ *are* **isomorphic,** *if there exists a bijection* $f : H \to H'$, *such that, for all* $x, y \in H$, $f(x \cdot y) = f(x) \cdot f(y)$, *where, as usual,* $f(A) = \{f(a) \mid a \in A\}$, *for* $\emptyset \neq A \subset H$.

Definition 2.3 *A hypergroup* H *is a* **D-hypergroup** *if there exists a group* G *and a subgroup* S *of it, such that* D *is isomorphic with* $G/\cong_l^S$, *the hypergroup of left cosets with respect to* S *in* G. (see Utumi [25]).

Definition 2.4 *A hypergroupoid* $(H, \cdot)$ *is a* **cogroup** *if there exist an element* $e \in H$ *and a mapping* $^{-1} : H \to \mathcal{P}(H)^*$, *such that, for all* $x, y, z \in H$:

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(ii) $e \cdot x = x$ *(left scalar identity),*
(iii) $y \in x^{-1} \Leftrightarrow e \in x \cdot y$,
(iv) $x \in y \cdot z \Rightarrow y \in x \cdot z^{-1} \wedge x \in y^{-1}x$,
(v) $x \cdot y \cap z \cdot y \neq \emptyset \Rightarrow x \in z \cdot e$,
(vi) $|x \cdot y| = |x \cdot z|$.

Remark 1 (i) Each D-hypergroup is a cogroup, while the converse slatement is not generally true.

(ii) In a cogroup, $x \in y \cdot e \Leftrightarrow x \cdot e = y \cdot e \Leftrightarrow xz = yz$, for all $z \in H$.

Theorem 1 (Krasner - Utumi). *A cogroup $\langle H, \cdot e\rangle$ is a D-hypergroup, if and only if there exists a group G of multipliers of H, i.e.*

$$G = \{\sigma \in S(H) | \sigma(x \cdot y) = \sigma(x) \cdot y,\ \forall x, y \in H\},$$

such that $H = G(e)$ and $S = Stab_G(e)$ verifies the condition $S(x) = x \cdot e$, $\forall x \in H$.

Definition 2.5 *A hypergroup $\langle H, \cdot\rangle$ which is commutative and satisfies the implication*

$$a/b \cap c/d \neq \emptyset \Rightarrow a \cdot d \cap b \cdot c \neq \emptyset,$$

where $a/b = \{x | a \in x \cdot b\}$, is called a **join space.**

Definition 2.6 *A nonempty subset S in H is a* **subhypergroup** *of H if for all $x \in S, x \cdot S = S \cdot x = S$.*

Definition 2.7 *A hypergroup $\langle H, \cdot\rangle$ is called a* **hypergroup of type U** *if it has a scalar identity e ($x \cdot e = e \cdot x = x$, for all $x \in H$) such that if $x \in x \cdot y$ for $x, y \in H$, then $y = e$.*

Definition 2.8 *The hypergroup is* **of type C** *if it has a scalar right identity e and the following implication holds: $x \cdot y \cap x \cdot z \neq \emptyset \Rightarrow e \cdot y = e \cdot z$. A hypergroup H of type C is a* **right cogroup,** *if, for all $x, y, z \in H, |x \cdot y| = |x \cdot z|$.*

Definition 2.9 *If the hypergroup $\langle H, \cdot\rangle$ is commutative ($x \cdot y = y \cdot x$, for all $x, y \in H$), has a scalar identity, e, every element x in H has a unique inverse (for $x \in H, x' \in H$ is an inverse, if $e \in x' \cdot x \cap x \cdot x'$) and it is reversible (i.e. $z \in x \cdot y$ implies $y \in z \cdot x'$), then H is a* **canonical hypergroup.**

A join space is a canonical hypergroup if and only if it has a scalar identity.

Usually we do not denote the hyperproduct of x and y by $x \cdot y$ but by xy (without " $\cdot$ ").

Definition 2.10 *A hypergroup $\langle H, \cdot\rangle$ with a scalar identity and unique inverses such that $x \in yz$ implies $y \in xz^{-1}$ and $z \in y^{-1}x$, for $x, y, z \in H$, is called a* **polygroup.**

We do not insist, but every type of hypergroup has appeared in connection with a nice example in algebra or finite geometries.

Definition 2.11 *Let A be a nonempty subset of a hypergroup $\langle H, \cdot \rangle$. The* **completion of A**, $C_H(A)$, *is obtained by the following construction:*

$$K_1(A) = A,\ K_{n+1}(A) = \{x \in H : \exists p \in \mathbb{N},\ \exists h_1, \dots h_p \in H,$$

$$x \in \prod_{i=1}^{p} h_i,\ (\prod_{i=1}^{p} h_i) \cap K_n \neq \emptyset\},\ n \geq 1.$$

Then $C_H(A) = \cup_{n \geq 1} K_n(A)$.

The subset A is called a **complete subset** *of the hypergroup H if $C_H(A) = A$.*

It is immediate that an intersection of complete parts of H is also a complete part of H, if it is nonempty.

Definition 2.12 *Let $\langle H, \cdot \rangle$ be a hypergroup and A be a subhypergroup in H. We say that A* **acts indistinguishably on H** *if: $xy = Ay$ and $yx = yA$, for all $x \in A, y \in H \setminus A$.*

A **consists of inseparable elements,** *if $xy \cap A \neq \emptyset$, for $x, y \in H \setminus A$, implies $A \subseteq x \cdot y$.*

J. Jantosciak [16] has shown that the notions have no relationships between them; there are examples of A acting indistinguishably but not consisting of inseparable elements and vice versa.

Examples (given by Jantosciak [16]) of these concepts on three hypergroups are the following ones, on the set $H = \{a, b, c\}$, and denoting $\{a, b\}$ by A:

$\cdot$	a	b	c
a	a	A	H
b	A	b	H
c	H	H	$\{a, c\}$

(I)

$\circ$	a	b	c
a	a	A	H
b	A	b	c
c	H	c	c

(II)

$*$	a	b	c
a	a	A	H
b	A	b	H
c	H	H	c

(III)

The subhypergroup A acts operationally equivalent but has not inseparable elements in (I), is constituted by inseparable elements which are not acting indistinguishably in (II) and satisfies both conditions in (III).

Definition 2.13 *An element $i \in H$, $\langle H, \cdot \rangle$ being a hypergroup, is called* **idempotent,** *if $i \cdot i = i$.*

It is known that the idempotents play their important in role semigroup theory and generally in algebra. In the hypergroup theory, the idempotents yield certain decompositions and constructions, as we shall see this in that follows.

Definition 2.14 *An equivalence ρ on the hypergroup $\langle H, \cdot \rangle$ is called* **strongly regular**, *if for all $x, y, a \in H$, the following implications hold:*

$$x\rho y \Rightarrow z\rho t, \ \forall z \in x \cdot a, \ \forall t \in y \cdot a,$$

and

$$x\rho y \Rightarrow u\rho v, \ \forall u \in a \cdot x, \ \forall v \in a \cdot y.$$

The smallest strongly regular equivalence on a hypergroup $\langle H, \cdot \rangle$ exists; it is denoted bu β^* and it has the following main property:

H/β^* may be endowed with a group operation induced by the hyperoperation on H, namely

$$\hat{x} \cdot \hat{y} = \hat{z},$$

where $\hat{x} = \{x' \in H | x'\beta^* x\}$, $\hat{y} = \{y' \in H | y'\beta^* y\}$ and $z \in x \cdot y$.

(See, for example, Corsini [5]: 11, 56, 57, 65.)

Moreover, β^* is the smallest equivalence for which H/β^* is a group and the mapping $\varphi_H : H \to H/\beta^*$, $\varphi_H(x) = \hat{x}$, is a homomorphism, i.e.

$$\varphi(x \cdot y) = \varphi_H(x) \cdot \varphi(y), \quad \text{for all } x, y \in H,$$

and $Ker\varphi_H = \omega_H$ is called the **heart** of the hypergroup H.

Theorem 2 *Let $\langle H, \cdot \rangle$ be a hypergroup, ω_H be its heart and A be a non-empty subset of H. Then:*

i) $\omega_H \cdot A = A \cdot \omega_H = \varphi_H^{-1}(\varphi_H(A))$;

ii) $\omega_H \cdot A = A \cdot \omega_H = C_H(A)$;

iii) *A is complete if and only if $A \cdot \omega_H = A$;*

iv) *ω_H is the intersection of all subhypergroups of H which are complete parts of H.*

This theorem is proved in [5], under the form of a sequence of theorems (66, 67, 74, 75).

3 Some constructions of hypergroups.

We recall here some constructions in hypergroup theory and their first properties, as well as some possibilities of getting hypergroups from other algebraic structures as groups, color schemes, projective geometries.

Many of these constructions can be viewed as extensions of a hypergroup by another hypergroup (group) or by a family of hypergroups.

Some constructions do not preserve important properties of the used hypergroups. For example, if H and G are polygroups, the (H,G) - hypergroup (see 3.2.) is not a polygroup.

For some ideas here it is necessary to consider again the papers introducing some classes of hypergroups and connecting hypergroups with graphs like [9], [10], [13], [15]. Looking at these constructions, it seems do not be so interesting to see what happens with group properties when passing to hypergroups. Some constructions are obviously inspired by other than "groupal" ideas.

We present then two more general constructions given by Jantosciak [16], and we make some comments on all constructions in hypergroup theory listed above.

Definition 3.1 *Let V be a finite set of vertices, C be a nonempty set with an involution $i : C \to C$ defined on it ($i \circ i = 1_C$) and $\{C_a, a \in C\}$ be a family of nonempty subsets of $V \times V \setminus \delta(V)$, where $\delta(V) = \{(v,v)|v \in V\}$. If the following axioms are fulfilled:*

(i) $\cup_{a \in C} C_a = V \times V \setminus \delta(V)$;

(ii) $C_a \cap C_b = \emptyset$, *if* $a \neq b$;

(iii) $C_{i(a)} = \{(x,y)|(y,x) \in C_a\}$;

(iv) $\forall x \in V,\ \forall a \in C,\ \exists y \in V,\ (x,y) \in C_a$;

(v) $\forall a,b,c \in C,\ C_c \cap (C_a/C_b) \neq \emptyset \Rightarrow C_c \subseteq C_a/C_b$, *where* $C_a/C_b = \{(x,z)|$ *if* $\exists y \in V,\ (x,y) \in C_a$ *and* $(y,z) \in C_b\}$;

then the couple $\langle V, (C_a)_{a \in C}\rangle$ is called a **color scheme.**

We may view C_a as the set of arrows in the complete oriented graph on V (without loops) which are colored by the colour a in C (set of colours).

The involution i guarantees that the colour of the arrow (y,x) depends only of the colour of the arrow (x,y).

Construction 3.12. Consider $M_V = \langle C \cup \{I\}, \cdot, ^{-1}, I\rangle$, where I is a symbol which does not belong to C. The **color algebra on V** is defined by:

(i) $a^{-1} = i(a)$, $\forall a \in C$, and $I^{-1} = I$;

(ii) $a \cdot I = I \cdot a = a$, $\forall a \in C \cup \{I\}$ and

$$a \cdot b = \begin{cases} \{c \in C | C_c \subseteq C_a | C_b\}, & \text{if } a \neq b^{-1}, \\ \{c \in C | C_c \subseteq |C_b\} \cup \{I\}, & \text{if } a = b^{-1} \end{cases} \quad \text{when } a,b \in C.$$

Then M_V is a polygroup associated to the given color scheme (see Comer [1,2]).

Definition 3.2 *A polygroup is called* **chromatic** *if it is isomorphic to the color algebra associated to a color scheme for a complete graph.*

The class of chromatic polygroups is denoted by **CHROM.**

Construction 3.2. The (H,G)-hypergroups have been introduced by M. De Salvo [7] and studied intensively by many mathematicians. Let $\langle G, \cdot, 1\rangle$ be a group, $\langle H, \circ\rangle$ be a hypergroup, $\{A_i, i \in G\}$ be a family of nonempty sets indexed by G such that $A_1 = H$ and $A_i \cap A_j = \emptyset$, if $i \neq j$, i and j in G. Then we define a hypergroup $\langle K, \otimes\rangle$ called the **(H,G)- hypergroup**, by taking $K = \cup_{i\in G} A_i$ and the hyperoperation $\otimes$ defined by:

$$x \otimes y := \begin{cases} x \circ y, \text{ if } & (x,y) \in A_1 \times A_1 \\ A_k, \text{ if } & (x,y) \in a_i \times A_j \ (i,j) \neq (1,1),\ i \cdot j = k. \end{cases}$$

Example 1 If $G = Z_2$, $A_{\hat{0}} = H$, $A_{\hat{1}} = A$, $A \cap H = \emptyset$, then $K = H \cup A$ and $x \otimes y = x \circ y$, if $(x,y) \in H \times H$, $x \otimes y = H$, if $(x,y) \in A \times A$ and $x \otimes y = y \otimes x = A$, if $(x,y) \in H \times A$. Here $\langle K, \otimes\rangle$ is a cyclic hypergroup generated by any $a \in A$; indeed, $a \otimes a = H$, $a \otimes h = A$ for any $h \in H$. Moreover, $\omega_K = H$. (See Corsini [5], Thm. 291.)

Proposition 1 *Let K be the (H, G) - hypergroup defined in 3.2. Then:*

i) K is commutative if and only if H and G are commutative (Corsini [5], Remark 368).

ii) If H is canonical and G is abelian, then $\langle K, \otimes\rangle$ is a canonical hypergroup (Corsini [5], Thms. 368, 369), *with the same scalar identity as H.*

iii) If H is a join space and G is abelian, then $\langle K, \otimes\rangle$ is a join space and conversely (Corsini [5], Thm. 370).

Construction 3.3. The **S-hypergroup** was defined in Vougiouklis [26] for studying very thin hypergroups. If $\Delta = \{S_i : i \in I\}$ is a nonempty family of pairwise disjoint hypergroups indexed by I, then $S = \cup_{i\in I} S_i$ may be equipped with the hypergroup hyperoperation:

$$x \otimes y = \begin{cases} x \cdot y, & \text{if } x, y \in S_i, \\ S_i \cup S_j, & \text{if } x \in S_i,\ y \in S_j,\ i \neq j. \end{cases}$$

Construction 3.4. Let $(G, \cdot)$ be a group, S be a subgroup. We may define the **polygroup of the double cosets of G with respect to S:**

$$\{SxS | x \in G\},$$

by using the hyperoperation:

$$SxS \circ SyS = \{SzS | z \in xSy\}.$$

The class of all polygroups isomorphic to such a polygroup is denoted by DB-COSET (Group).

Construction 3.5. Wreath product of two polygroups. This was defined by S.D. Comer [2] and the idea came from graph theory. Let A and B be two polygroups, $A \cap B = \emptyset$ or $\{e\}$, where e is the scalar identity.

We consider $A[B] := (B \setminus \{e_B\}) \cup A$ and the canonical map $\sigma : A[B] \to B$, defined by $\sigma(a) = e_B$, for all $a \in A$ and $\sigma(x) = x$, for all $x \in B \setminus \{e\}$. A[B] is a polygroup with respect to the following hyperoperation:

$$x \otimes y = \begin{cases} x \cdot y, & \text{if } x, y \in A, \\ \sigma^{-1}(\sigma(x) \cdot \sigma(y)), & \text{otherwise.} \end{cases}$$

In fact, the hyperoperation is the following one:

$$x \otimes y = \begin{cases} x \cdot y, & \text{if } x, y \in A, \\ x, & \text{if } x \in B, y \in A, \\ y, & \text{if } x \in A, y \in B, \\ x \cdot y, & \text{if } x, y \in B, y \neq x^{-1} \\ x \cdot y \cup A, & \text{if } x, y \in B, y = x^{-1}. \end{cases}$$

Proposition 2 *The extension A[B] is a polygroup which preserves the property of being chromatic and* $A[B]/\beta^* \simeq B/\beta^*$.

Construction 3.6. Let $\langle G, \cdot \rangle$ be a group (resp. a polygroup) and θ be an equivalence on G which satisfies the following conditions:

(i) $x\theta y \Rightarrow x^{-1}\theta y^{-1}$;

(ii) $z'\theta z$ and $z = x \cdot y$ (resp. $z \in x \cdot y$) $\Rightarrow$ there exist $x', y' \in G$, such that $x\theta x'$, $y\theta y'$ and $z' = x' \cdot y'$ (resp. $z' \in x' \cdot y'$);

(iii) $x\theta e \Rightarrow x = e$.

Then $\langle G/\theta, *, \hat{e} \rangle$, with $\hat{x} * \hat{y} = \{\hat{z} | z = a \cdot b$ (resp. $z \in a \cdot b$), with $a\theta x$ and $b\theta y\}$, is the **double quotient polygroup induced by θ on G.** The class of those polygroups isomorphic to a double quotient polygroup associated to a group is denoted by Q^2 **(Group).**

Theorem 3 (Comer [4]). *The classes of polygroups denoted by* **CHROM, Q^2 (Group), DBCOSET (Group)** *are all closed under wreath product.*

Construction 3.7. Enlargement and completion of a hypergroup (See Corsini [5], Definition 373): A family of pairwise disjoint nonempty sets $\{A(x) : x \in H\}$ indexed by H, where $\langle H, \circ \rangle$ is a hypergroup, is used in this construction. We consider the set $K_H := \cup_{x \in H} A(x)$, and we define a function $g : K_H \to H$, by putting $g(a) = x$, when $a \in A(x)$. The hypergroup hyperoperation is given by

$$a \square b = \cup_{x \in g(a) \circ g(b)} A(z), \text{ for all } a, b \in K_H.$$

In this way, we obtain $\langle K_H, \square \rangle$, the **enlargement of H,** called also the K_H **- hypergroup** (see De Salvo [8]).

Now, the **completion of the hypergroup** $\langle H, \circ \rangle$ is the hypergroup $\Delta(H) := H$, with the hyperoperation given by:

$$x \otimes y = C_H(x \circ y), \text{ for all } x, y \in \Delta(H).$$

Theorem 4 (Corsini [5], Thms. 384, 384). *Let $\langle H, \circ \rangle$ be a hypergroup, $\{A(x) : x \in H\}$ be a family of nonempty pairwise disjoint sets. Then:*

i) $\Delta(K_H) = K_{\Delta(H)}$;

ii) $\omega_{K_H} = \omega_{\Delta(K_H)} = \omega_{K_{\Delta(H)}}$.

Construction 3. 8. Lexicographic product of two hypergroups $\langle H_i, \circ \rangle$, $i = 1, 2$:

On the set $H_1 \times H_2$, if H_1 has only idempotent elements, and we denote (x_1, x_2) by $x_1 \times x_2$, and by $x_1 \times A_2$ we mean $\{(x_1, a_2) : a_2 \in A_2\}$, where $A_2 \subset H_2$ (and so on), we may define the following hypergroup structure:

$$(x_1, x_2) \circ (y_1, y_2) := \begin{cases} (x_1 \circ y_1) \times H_2, & \text{if} \quad x_1 \neq y_1, \\ x_1 \times (x_2 \circ y_2), & \text{if} \quad x_1 = y_1. \end{cases}$$

Then $\langle H_1 \times H_2, \circ \rangle$ is a hypergroup, **the lexicographic product of** H_1 **and** H_2 (see, for more, [11]).

Construction 3.9. Lexicographic sum of a family of subgroups:

Let $\langle Q, \circ \rangle$ be a hypergroup with all elements idempotent, $\Delta = \{H_i : i \in Q\}$ be a family of pairwise disjoint hypergroups. $S = \cup_{i \in Q} H_i$, with the hyperoperation defined by:

$$x \cdot y = \begin{cases} xy, & \text{if} \quad i = j \\ \cup\{H_k : k \in i \circ j\}, & \text{if} \quad i \neq j, \end{cases}$$

where $x \in H_i$, $y \in H_j$, is a hypergroup, the **lexicographic sum** of the given family.

Remark 2 If the hypergroups H_i are not pairwise disjoint, then it is required the property: $\{i, j\} \subseteq i \circ j$, if $i \neq j$ in Q.

Construction 3.10. There exists also **standard direct product of hypergroups.**

It is easily seen the following

Theorem 5 *The classes of all polygroups,* **CHROM,** Q^2 **(Group) and DB-COSET (Group)** *are closed under the formation of finite direct products* (see Comer [4]).

Construction 3.11. Extension of a hypergroup A by a group B (the semidirect product $A \times_\tau B$ **with respect to a group homomorphism** $\tau : B \to Aut(A)$) : Sureau [23] has studied first this construction with its properties, but, in [27], Vougiouklis has introduced the semidirect product $A \times_\tau B$ of two hypergroups A and B with respect to a group homomorphism $\tau : B/\beta_B^* \to Aut(A)$.

The semidirect product $A \times B$ has the following hyperoperation (its underlying being set $A \times B$) :

$$a_1, b_1) \cdot (a_2, b_2) = \{(x, y) \in A \times B : x \in a_1 \cdot \tau(b_1)(a_2),\ y \in b_1 \cdot b_2\} =$$

$$= (a_1 \cdot \tau(b_1)(a_2)) \times (b_1 \cdot b_2).$$

(The last equality gives a possible simple notation.)

Construction 3.12. Consider the **projective geometry** $G = \langle P, L \rangle$, where P is a nonempty set of points, $L \subseteq \mathcal{P}(P)^*$ is the set of lines, such that:

(i) Each line in L has at least three distinct points;

(ii) For each pair of distinct points $p, q \in P$, there exists a unique line $\overline{pq} \in L$ containing p,q;

(iii) For all $p, q, r \in P$, distinct points, $\overline{pq} \cap \overline{rs} \neq \emptyset \Rightarrow \overline{ps} \cap \overline{qr} \neq \emptyset$.

Then there exists an **associated polygroup,** $\mathcal{P}_G = \langle P \cup \{i\}, *, i, ^{-1} \rangle$, with $i \notin P$, where: $p^{-1} = p$, $i^{-1} = i$, for all $p \in P$, and

$$p * q = \begin{cases} p, & \text{if } q = i; \\ q, & \text{if } p = i; \\ \{i, p\}, & \text{if } p = q \neq i; \\ \overline{pq} \setminus \{p, q\}, & \text{if } p \neq q, p, q \in P, \end{cases}$$

for $p, q \in P \cup \{i\}$.

4 Constructions using idempotents

Let $\langle H, \cdot \rangle$ be a hypergroup, I be a subset of idempotent elements of H. Consider a family of pairwise disjoint hypergroups indexed by I, $\Delta = \{A_i | i \in I\}$, such that $A_i \cap (H \setminus I) = \emptyset$, for all $i \in I$.

Proposition 3 *(i) With the above hypotheses and notations, the set* $K = (H \setminus I) \cup (\cup_{i \in I} A_i)$ *is a hypergroup with respect to the following hyperoperation:*

$$x \circ y = \begin{cases} x \cdot y, & \text{if } x, y \in A, \text{ for an } i \in I, \\ \sigma^{-1}(\sigma(x) \cdot \sigma(y)), & \text{otherwise,} \end{cases}$$

for $x, y \in K$, *where* $\sigma : K \to H$ *is given by*

$$\sigma(x) = \begin{cases} i, & \text{if } x \in a_i, \text{ for an } i \in I, \\ x, & \text{if } x \in H \setminus I, \end{cases}$$

for all $x \in K$.

(ii) The function σ *is a hypergroup homomorphism (i.e.* $\sigma(x \circ y) = \sigma(x) \cdot \sigma(y)$, *for all x,y in K).*

(iii) Each $A_i, i \in I$, *is a subhypergroup of K consisting of inseparable elements and acting indistinguishably on K.*

Proof. It is already done Jantosciak [16], in proofs of the Propositions 3, 4 and 5.

Jantosciak denoted this hypergroup by $\Delta_I(H)$, calling σ the **canonical homomorphism** of $\Delta_I(H)$ onto H and the construction is called the **expansion of H by the hypergroups of Δ through the idempotents in I.**

Now, we consider the case of a hypergroup $\langle K, \cdot \rangle$ with a family $\Delta = \{A_i | i \in I\}$ of subhypergroups indexed by a nonempty set I, such that $A_i \cap A_j = \emptyset$, for all $i \neq j$; then

$$P = \{A_i | i \in I\} \cup \{\{x\} | x \in K \backslash (\cup_{i \in I} A_i)\}$$

is a partition of K and the associated equivalence ρ on K gives a factor-set K/ρ. We denote the class x by x, if $x \in K\backslash(\cup_{i\in I}A_i)$, and we denote by i the class $A_i, i \in I$.

We consider A_i acting indistinguishably on K, for all $i \in I$, and being composed of inseparable elements.

Now, we denote by x_ρ the class with respect to ρ represented by $x \in K$. So $x_\rho = A_i$ if $x \in A_i$, we denote also it by i, and $x_\rho = x$, if $x \in K\backslash(\cup_{i\in I}A_i)$.

Define hypercomposition between two classes:

$$x_\rho * y_\rho = \{z_\rho | z \in x_\rho \cdot y_\rho\}.$$

This means $i*i = i$ (if there exists i such that $x, y \in A_i$, that is $x_\rho = y_\rho = A_i = i$; then $A_i \cdot A_i \subseteq A_i$, such that $A_i * A_i = A_i$).

In the other cases, if $x \in A_i$ and $y \notin A_i$, we have $x_\rho \cdot y_\rho = A_i \cdot y_\rho = x \cdot y_\rho$, since A_i acts indistinguishably on $K\backslash A_i$.

Now, if $y \in A_j,\ j \neq i,\ x \cdot y_\rho = x \cdot A_j = x \cdot y$, therefore

$$x_\rho * y_\rho = i * y = \{z_\rho | z \in x \cdot y\}.$$

If $x \in A_i$ and $y \in K\backslash(\cup_{i\in I}A_i)$, then

$$x_\rho * y_\rho = A_i * y = \{z_\rho | z \in A_i \cdot y = x \cdot y\} = \{z_\rho | z \in x \cdot y\}.$$

But, if $x \notin A_k$ or $y \notin A_k$ and $A_k \cap x \cdot y \neq \emptyset$; we have $A_k \subseteq x \cdot y$ we may have also $z_\rho = \{z\} \subseteq x \cdot y$. This shows that $x \cdot y$ is a union of classes with respect to ρ, therefore we may write:

$$(x \cdot y)/(\rho \cap x \cdot y) \subseteq K/\rho.$$

We have then, for all $x_\rho, y_\rho \in K/\rho$, the hyperoperation:

$$x_\rho * y_\rho = \begin{cases} i, & \text{if } x, y \in A_i \text{ for an } i \in I, \\ (x \cdot y)/(\rho \cap x \cdot y), & \text{otherwise.} \end{cases} \tag{8}$$

Proposition 4 *Let $\langle K, \cdot \rangle$ be a hypergroup, $\Delta = \{A_i | i \in I\}$ be a nonempty family of subhypergroups such that:*

i) $A_i \cap A_j = 0$, if $i \neq j$, $i, j \in I$;

ii) $A_j \cap (K \setminus \cup_{i=1} A_i) = \emptyset$, for all $j \in I$;

iii) A_i acts indistinguishably on K and it is composed of inseparable elements, for all $i \in I$.

Let ρ be the equivalence on K corresponding to the partition P from above. Then $\langle K/\rho, * \rangle$, with "$*$" given by (3.2.) is a hypergroup in which $\{i(= A_i) | i \in I\}$ is a set of idempotent elements.

Proof. We have: $(x_\rho * y_\rho) * z_\rho = \{t_\rho | t \in x \cdot y\} * z_\rho = \cup_{t \in x \cdot y} (t_\rho * z_\rho) = \cup_{t \in x \cdot y} \{w_\rho | w \in t \cdot z\} = \{w_\rho | w \in (x \cdot y) \cdot z\} = \{w_\rho | w \in x \cdot (y \cdot z)\} = \cup_{t \in y \cdot z} \{w_\rho | w \in x \cdot t\} = \cup_{t \in y \cdot z} (x_\rho * t_\rho) = x_\rho * \{t_\rho | t \in y \cdot z\} = x_\rho * (y_\rho * z_\rho)$.

Then $x_\rho * (K/\rho) = \{z_\rho | z \in x_\rho \cdot K\} = \{z_\rho | z \in K\} = K/\rho$ and, analogously, $(K/\rho) * x_\rho = K/\rho$.

Could we reobtain $\langle K, \cdot \rangle$ from $\langle K/\rho, * \rangle$? Not exactly; we obtain the initial hyperoperation for the elements $x, y \in K$, for which there is not an $i \in I$ with $x, y \in A_i$, namely:

$$x \circ y \cup \{x_j | z_j \in x_j * y_j\} = x \cdot y.$$

If there exists $i \in I$ such that $x, y \in A_i$, we do not have $x \circ y = x \cdot y$. Defining $x \circ y$ as above, we have $x \circ y = A_i$.

Therefore we have:

Proposition 5 *The hyperoperation on K from above, defined by:*

$$x \circ y = \begin{cases} A_i, & \text{if there exists } i \in I, x, y \in A_i, \\ x \cdot y, & \text{otherwise} \end{cases}$$

is a hypergroup composition on K, and it is different from the former hyperoperation only for pairs of elements belonging to the same class A_i.

These two constructions give new hypergroups and they can be used for expressing many constructions given in Section 3. (See [16], Section 6.)

a) For the (H, G) - hypergroup in 3.2., we have the expansion of the group G by the hypergroup H through the identity 1 of G. Namely, $I = \{1\}$, $\Delta = H$, $\Delta_I(G) = H_1(G)$ is an (H,G) - hypergroup.

b) The wreath product of two polygroups A[B] (in 3.9.) is the expansion of B by A through the identity e of B; namely, $I = \{e\}$, $\Delta = A$, $\Delta_I(B) = A_e(B) = A[B]$.

c) The S-hypergroup in Construction 3.3. has the family $\Delta = \{S_i | i \in I\}$ and it is $\langle \Delta_I(I), \circ \rangle$, where I is a hypergroup with respect to the set union: $i \cup j = \{i, j\}$.

d) The lexicographic product $H_1 \times H_2$ (in Consteuction 3.8) is the expansion of H_1 through idempotents, where: $I = H_1$, $\Delta = \{x_1 \times H_2 | x_1 \in I\}$ and $\Delta_I(H_1)$=$(H_1 - H_1) \cup (\cup\Delta)$= $H_1 \times H_2$, $\sigma(x) = x_1$ for $x = (x_1, x_2)$, and $x \circ y = c \cdot y$, $\forall x, y \in H_1 \times H_2$.

e) Lexicographic sum (in Construction 3.9) is the expansion of Q by Δ through Q, namely: $\Delta_Q(Q) = \cup\Delta$, $\sigma(x) = i$ for $x \in H_i$ and $x \circ y = x \cdot y$ for $x, y \in \Delta_Q(Q)$.

We recall now a construction of a cogroup described by M. Guţan in [12] and inspired by Sureau [23] and Haddad and Sureau [14].

Construction.

Let $\langle G, \cdot \rangle$ be a group, A be a nonempty set such that $A \cap G = \emptyset$ and $\varphi : G \to A$ be a bijection. Then we may define a hyperoperation on $K = G \cup A$, by taking:

$$x \circ y = \begin{cases} x \cdot y, & \text{if } (x,y) \in G \times G, \\ G, & \text{if } (x,y) \in A \times A, \\ A, & \text{if } (x,y) \in G \times A, \\ \varphi(\varphi^{-1}(x) \cdot y, & \text{if } (x,y) \in A \times G, \end{cases}$$

and $\langle K, \circ \rangle$ is a cogroup.

If $|G| = n$, then the cogroup K is of the type $\mathcal{H}(1, 1, \ldots, 1, n)$, where 1 is taken for n times and the above construction characterizes such a cogroup (see Guţan [12], Thm. II.2.1).

5 Open questions

After reviewing some constructions in hypergroup theory, we recall some problems posed by S. Comer [4] in 1990, which remained remained yet unsolved

1. *Determine all polygroup extensions P of a polygroup A by a polgroup B. do the same for other classes of hypergroups:* Q^2 (Group), DBCOSET (Group) or CHROM.

This is in connection with the wreath product A[B] which can be interpreted in the language of the first two constructions in Section 4. Such extensions were studied by Sureau [24] and Vougiouklis [27].

2. *Denote by EXT(A,B) the class of all isomorphism types of extensions of A by B. Does EXT(A,B) have any type of algebraic structure?*

We recall that such an algebraic structure (a group structure) allowed us to use the cohomology group of order 2 instead of Ext(A,B) in case of two groups, A being abelian.

3. *Develop a reasonable decomposition theory for special classes of polygroups where the factors cannot be decomposed with respect to the direct (wreath or semidirect) product.*

Since S. Comer stated these problems, there is not a significant step in solving them, but in the case of finite hypergroups there are some results of decomposition type and equivalences on hypergroups were much studied.

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ALGEBRAS GROUPS AND GEOMETRIES 38 115 - 121 (2022) - 115 -
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ON *W*-HYPERGROUPS

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Abstract

In this note one gives some examples and some propositions which can be useful to construct W−hypergroups.

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Introduction.

A W–hyperoperation on a non empty set H is a function $< \otimes >$ which associates to every pair $(x, y) \in H \times H$ a multisubset of H, i.e., a subset $x \circ y$ where every element z appears with a multiplicity $m_{(x,y)}(z) \geq 1$. The W–hypergroups introduced in a particular case by Wall in [7] have not been, till now, cultivated so much as Marty hypergroups. Particular cases of them, endowed with a topology have been utilized in harmonic analysis, measure and probability theory. They have been considered again by Corsini–Mittas [2] and recently by Rosenberg (see these Proceedings). They deserve by our opinion to be studied throughly both for their theoretical interest and for applications to combinatorics, analysis, particle physics.

§1. **Definition 1.** If $\{A_i\}_{i \in I}$ is a family of nonempty sets let $[A_i]_{i \in I}$ (in the finite case also, $[A_1, A_2, ..., A_n]$) be the multiset which has $\bigcup_{i \in I} A_i$ as support and such that for any $u \in \bigcup_{i \in I} A_i$, its multiplicity is $|\{i \mid A_i \ni x\}|$.

Definition 2. Let A be a multiset, set $\hat{A}$ the set of elements x such that $m_A(x) > 0$ for any $x \in \hat{A}$, let $m_A(x)$ be the multiplicity of x in A. If A, B are multisets, set

$$A \prec B \quad \text{iff} \quad \forall x \in A,\ m_A(x) \leq m_B(x).$$

Let $A \vee B$ be the multiset on $\hat{A} \cup \hat{B}$ defined

$$\forall x \in \hat{A} \cup \hat{B},\ m_{A \cup B}(x) = \max\{m_A(x), m_B(x)\}.$$

Let $A \cap B$ be the multiset on $\hat{A} \cap \hat{B}$ defined

$$\forall x \in \hat{A} \cap \hat{B},\ m_{A \cap B}(x) = \min\{m_A(x), m_B(x)\}.$$

If A is a multiset and $x \in \hat{A}$ let us denote $[x, x]$ or $[x]_2, ..., [x_1, x_2, ..., x_n]$ where $\forall i,\ x_i = x$, or $[x]_m$ the multiset on $\{x\}$ with $m_{\hat{x}}(x) = n$.
In a similar way we shall denote by $[x, ..., x_s, v, ..., v_t, z, ..., z_u, ...]$ or $[x_s, v_t, z_u, ...]$ the multiset where $m_Q(x) = s, m_Q(v) = t, m_Q(z) = u, ...$ and $Q = \{x, v, z, ...\}$.

Let $< G; + >$ be an abelian group.

Let R be the equivalence on G such that $\forall n \in G$, $R(n) = \{x, -x\}$[1].
In G/R set $\bar{x} \circ \bar{y} = \{\overline{x+y}, \overline{x-y}\}$. G/R is a hypergroup (see [1]).

Theorem 1. *Set* $\bar{x}\square\bar{y} = [\overline{x+y}, \overline{x-y}]$ *that is:*
if $|\bar{x} \circ \bar{y}| = 2$, *then* $\bar{x}\square\bar{y} = \bar{x} \circ \bar{y}$;
if $|\bar{x} \circ \bar{y}| = 1$, *then* $\bar{x}\square\bar{y} = [\overline{x+y}, \overline{x+y}]$
Endowed with the W*-hyperoperation* $< \square >$, G/R *has a structure of* W*-hypergroup.*

Easy.

Let now $G = \boldsymbol{Z}/n\boldsymbol{Z}$. Let's look for the conditions on x, y such that $|x \circ y| = 1$.

Set $n = 2k$. We have $\bar{\imath}\square\bar{\jmath} = [\overline{\imath-\jmath}, \overline{\imath+\jmath}]$ where we suppose $k \geq \imath \geq \jmath$.

If $\imath+\jmath \geq k$, from $\imath+\jmath = k+t$, where $t \leq k$, we obtain $\overline{\imath+\jmath} = \overline{n-\imath-\jmath} = \overline{k-t}$ and $\imath = k+t-\jmath$.

If we set $\overline{\imath-\jmath} = \overline{\imath+\jmath}$, it follows $\imath - \jmath = k-t$ whence $k+t-2\jmath = k-t$, hence $2\jmath = 2t$, i.e., $\jmath = t$, therefore $\imath = k$.

If $\imath + \jmath < k$, then $\overline{\imath+\jmath} = \overline{\imath-\jmath}$ iff $\imath + \jmath = \imath - \jmath \Longleftrightarrow \jmath = 0$.

Set $n = 2k+1$. Let us suppose $\imath + \jmath = k + t$. In this case, $\overline{\imath+\jmath} = \overline{k+t} = \overline{\imath-\jmath}$ implies $2k+1-k-t = \imath-\jmath$ whence $k+1-t = k+t-2\jmath$ then $2\jmath = 2t+1$ which is absurd.

Therefore if $n = 2k+1$, $\overline{\imath+\jmath} = \overline{\imath-\jmath}$ iff $\jmath = 0$.

Let us suppose now $< G : + >$ to be a canonical hypergroup, let us define the relation R in the same manner as above, and set $\bar{x} \otimes \bar{y} = [\overline{x+y}, \overline{x-y}]$ (see Th. 1) for instance if $G =< C(4); \circ >$ is the hypergroup

$G =$

$\circ$	0	1	2
0	1	1	2
1	1	0,2	1
2	2	1	0

we obtain

[1]If R is an equivalence defined on a set A, we denote by $R(x)$ the subset $R(x) = \{y \in A \mid yRx\}$, and $\bar{x}$ the class (element of A/R) determined by x.

the W-hypergroup $< G; \otimes >=$

$\otimes$	0	1	2
0	0,0	1,1	2,2
1		0,0 2,2	1,1
2			0,0

§2. Let $< H; \circ >$ be a hypergroup such that

(I) $$\forall (x, y) \in H^2, \ |x \circ y| \leq 2$$

Let

$$\begin{aligned} A(H) &= \{(x, y) \in H^2 \mid |x \circ y| = 2\} \\ A'(H) &= H \times H - A(H). \end{aligned}$$

Set $< H; \otimes >$ the W–hypergroupoid defined

$$\begin{aligned} \forall (x, y) \ &\in A(H), \quad x \otimes y = x \circ y \\ \forall (x, y) \ &\in A'(H), \quad x \otimes y = [x \circ y, x \circ y]. \end{aligned}$$

Theorem 2. *If $< H; \circ >$ is a complete hypergroup satisfying* (I), *then $< H; \otimes >$ is a W–hypergroup.*

It's clear that it is enough to prove the associativity.

If $|(x \circ y) \circ z| \in \{1, 4\}$, the associativity is verified in any hypergroup satisfying (I).

Let us suppose $|(x \circ y) \circ z| > 1$. If $|(x \circ y)| = 2$, set $x \circ y = \{u_1, v_1\}$, then $(x \otimes y) \otimes z = [u_1 \circ z, v_1 \circ z]$, whence by the hypothesis of completeness, there are a, b in H such that $a \neq b$ and we have $u_1 \circ z = v_1 \circ z = \{a, b\}$. So, $(x \circ y) \circ z = u_1 \circ z = v_1 \circ z = x \circ (y \circ z) = \{a, b\}$. Then $(x \otimes y) \otimes z = [a, a, b, b]$. On the other side, by a similar argument one proves that there are u_2, v_2 in $y \circ z$ such that $x \otimes (y \otimes z) = [x \circ u_2, x \circ v_2]$. Whence $x \otimes (y \otimes z) = [a, a, b, b]$.

Theorem 3. *Let $< H; \circ >$ be a hypergroup satisfying* (I). *A necessary condition so that $< H; \otimes >$ may be a W-hypergroup is there is not a pair $(a, b) \in H^2$ such that*

(α) $$a \circ a = a \quad \textit{and} \quad a \circ b = \{a, b\}.$$

Indeed, $(a\otimes a)\otimes b = [a,a]\otimes b = [a\circ b]_2 = [a]_2 \vee [b]_2$, $a\otimes(a\otimes b) = a\otimes\{a,b\} =$
$= a\otimes a \vee a\otimes b = a_2 \vee [a,b] = [a]_3 \vee b$.

Remark. There are (I)–hypergroups which do not satisfy (α), are not complete and are such that $< H;\otimes >$ is a W–hypergroup. See the example H nr. 217 [1].

§3. Let $< H;\circ >$ be a hypergroup such that

$$(0)\quad \begin{cases} \forall a\in H, \text{ there is at most one element } b' \text{ and one element } b'' \\ \qquad\qquad\qquad\qquad \text{such that } |a\circ b'| = 2 = |b'\circ a|. \\ \forall x\in H : x\neq b', \quad |a\circ x| = 1 \\ \forall y\in H : y\neq b'', \quad |y\circ a| = 1 \end{cases}$$

Theorem 4. *Let us suppose H is a hypergroup satisfying* (0) *and moreover one of the following:*

(*) $\forall (x,y,z)\in H^3$, $|x\circ y\circ z| < 3$.

(**) *If* $(x,y,z)\in H^3$, $|x\circ y| = |y\circ z| = 2$ *and* $x\circ y\cap y\circ z\neq\emptyset$, *it follows* $x\circ y = y\circ z$.

Then the W–hypergroupoid $< H;\otimes >$ is a W–hypergroup.

Indeed, $\forall (a,b,c)\in H^3$, we have $(a\circ b)\circ c = a\circ(b\circ c)$.

(a_1) If $|a\circ b\circ c|\in\{1,4\}$ then it is clear that $<\otimes>$ is associative. Since if $|a\circ b\circ c| = 1$, $(a\circ b)\circ c = z$. We have

$$(a\otimes b)\otimes c = a\otimes(b\otimes c) = [z]_4$$

If $a\circ b\circ c = \{u_1,u_2,u_3,u_4\}$, where $\iota\neq\jmath \implies u_\iota\neq u_\jmath$ we have $(a\otimes b)\otimes c = a\otimes(b\otimes c) = [u_1,u_2,u_3,u_4]$.

Therefore we can suppose up to now $|a\circ b\circ c|\in\{2,3\}$.

(a_2) Let us suppose now $|a \circ b \circ c| = 2$. If $|a \circ b| = 1$ i.e. $\exists q : a \circ b = q$, then $(a \circ b) \circ c) = q \circ c = \{s, t\} = a \circ (b \circ c)$, $s \neq t$ and $(a \otimes b) \otimes c = = [q]_2 \otimes c = [q \otimes c] \vee [q \otimes c] = [s, t] \vee [s, t] = [s_2, t_2]$.

If $|b \circ c| = 1$, i.e. $\exists p \in H : b \circ c = p$, then $a \circ (b \circ c) = a \circ p = \{s, t\}$ and $a \otimes (b \otimes c) = a \otimes [p]_2 = [a \otimes p] \vee [a \otimes p] = [s]_2 \vee [t]_2 = [s_2, t_2]$.

If $|b \circ c| = 2$ i.e. $\exists (p_1, p_2)$, $p_1 \neq p_2 : b \circ c = \{p_1, p_2\}$ we have $a \circ (b \circ c) = = a \circ p_1 \cup a \circ p_2$, whence $a \otimes (b \otimes c) = a \otimes [p_1, p_2] = [a \otimes p_1, a \otimes p_2] = [s_2, t_2]$.

If $|a \circ b| = 2$, then $\exists (u_1, u_2) \in H^2$, $u_1 \neq u_2 : a \circ b = \{u_1, u_2\}$ and $u_1 \circ c = s$, $u_2 \circ c = t$, therefore $(a \otimes b) \otimes c = [u_1, u_2] \otimes c = [s_2, t_2]$.

(a_3) Let $|a \circ b \circ c| = 3$. This hypothesis implies $|a \circ b| = 2 = |b \circ c|$. Set $a \circ b = \{p, q\}$ $b \circ c = \{u, v\}$, $a \circ b \circ c = \{\lambda, \mu, \nu\}$. We have $(a \circ b) \circ c = = p \circ c \cup q \circ c$. We can suppose $p \circ c = \{\lambda, \mu\}$, $q \circ c = \nu$, hence $p = b$, from which $b \circ c = p \circ c = \{\lambda, \mu\}$, and there are u, v such that $a \circ (b \circ c) = a \circ u \cup a \circ v$, set $u = \lambda$, $v = \mu$. We can suppose $|a \circ u| = 2$, $a \circ v = \{\lambda, \mu, \nu\} - a \circ u$. From $|a \circ \lambda| = |a \circ u| = 2 = |a \circ b|$, it follows that $\lambda = u = b = p$.

There are three eventualities:

1) $a \circ \lambda = \{\lambda, \mu\}$ from which $a \circ \mu = \nu$.

2) $a \circ \lambda = \{\lambda, \nu\}$ from which $a \circ \mu = \mu$.

3) $a \circ \lambda = \{\mu, \nu\}$ from which $a \circ \mu = \lambda$.

From $\lambda = b$, it follows $a \circ \lambda = a \circ b = \{\lambda, q\}$. Then 3) is excluded.

Let's suppose $a \circ \lambda = a \circ b = \{\lambda, \nu\} = \{p, q\}$.

$\lambda = p$ implies $\nu = q$ from which $\nu \circ c = \nu$.

We have $b \circ c = \{\lambda, \mu\}$, therefore by the hypothesis $(*)$, $a \circ b = \{\lambda, \nu\}$, $b \circ c = \{\lambda, \mu\}$ implies $\mu = \nu$ which is absurd.

Then only 1) can be verified.

We have $(a \otimes b) \otimes c = [p \otimes c, q \otimes c] = [\lambda, \mu, \nu\nu]$

$a \otimes (b \otimes c) = a \otimes [\lambda, \mu] = [a \otimes \lambda, a \otimes \mu] = [\lambda, \mu, \nu\nu]$.

Remark. If $< H; \circ]$ is commutative, we have $b \circ c = c \circ b$, whence, since $|a \circ b| = |b \circ c| = 2 = |c \circ b|$, follows $a = c$.

Therefore $(a \otimes b) \otimes c = (a \otimes b) \otimes a$

$a \otimes (b \otimes c) = a \otimes (a \otimes b) = (a \otimes b) \otimes a.$

Remark that if $< \circ >$ is commutative, (x, y, z) does not exist such that $|x \circ y \circ z| = 3$ and $|\{x, y, z\}| > 1$. Indeed, $(a \circ b) \circ c = (a \circ b) \circ a = (a \circ a) \circ b$. Since $|a \circ b| = 2$, if $a \neq b$, we have $|a \circ a| = 1$, whence, setting $a \circ a = z$, we have $a \circ b \circ c = z \circ b$ and $|z \circ b| \leq 2$, therefore $a = b = c$.

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SOME GEOMETRICAL P-H_V-STRUCTURES

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Abstract

Using non-commutative hyperoperations in the plane and some subsets of this plane, we construct various P-H_v-structures .

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1. INTRODUCTION

In 1990, larger classes than the known hyperstructures were defined [4], the H_v-structures, where the classical axioms were replaced by the weak ones.
The hypergroupoid $\langle H, \cdot \rangle$ is called a **H_v-group** if $(\cdot)$ is weak associative and the reproduction axiom is valid.

Let $\langle G, \cdot \rangle$ be a semigroup and $P \subset G$, $P \neq \emptyset$. The following hyperoperations can be defined and they are called **P-hyperoperations** :

$$P^* \ : \ xP^*y = xPy$$

$$P^*_r \ : \ xP^*_r y = xyP \qquad \text{for all } x,y \in G.$$

$$P^*_\ell \ : \ xP^*_\ell y = Pxy$$

The P-hyperstructures were introduced for hypergroups in [5] and generalized in [6].
In this paper we deal with P^*_r hyperoperation which is defined in H_v-groups. A similar, but single-valued structure to which a multivalued one is associated, was introduced in [1].
In addition, we study the existence of units and inverses of these constructions and we connect them with join spaces [3].

2. DEFINITIONS

On the set $\mathbb{R}^2$ fix a point O. We define the following non-commutative hyperoperation :

For all x,y in $\mathbb{R}^2$, $\quad x \circ y = [x, x+y] = \{ x+\lambda y \ / \ \lambda \in [0,1] \}$.

Geometrically, $(\circ)$ gives the side x , $x+y$ of the parallelogram with vertices O , x , $x+y$, y (Appendix, fig.1).

Remarked that, $x \circ y \ni x$ for all $x,y \in \mathbb{R}^2$.

Then the hyperstructure $\langle \mathbb{R}^2, \circ \rangle$ is H_v-group [2]. It is easy to see that in this H_v-group the β^*-classes are equal to $\mathbb{R}^2$, i.e. the fundamental group consists of only one element.

Now, let P be the set $P = [O,p] = \{ \mu p \ / \ \mu \in [0,1] \} \subset \mathbb{R}^2$ where p is a fixed point of the plane.

Consider for $x,y \in \mathbb{R}^2$ the P-hyperoperation P^*_r :

$$P^*_r : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R}^2) \quad xP^*_r y = (x \circ y) \circ P \ .$$

We get that

$$xP^*_r y = [x, x+y] \circ [0, p] = \{ x+\lambda y \ / \ \lambda \in [0,1] \} \circ \{ \nu p \ / \ \nu \in [0,1] \} =$$

$$= \bigcup_{\lambda, \nu \in [0,1]} [x+\lambda y \ , \ x+\lambda y+\nu p] = \bigcup_{\lambda, \nu \in [0,1]} \{x+\lambda y+\tau \nu p \ / \ \tau \in [0,1]\}.$$

Finally, $xP^*_r y = \{ x+\lambda y+\mu p \ / \ \lambda, \mu \in [0,1] \}$.

Geometrically, $xP^*_r y$ is the closed parallelogram region with vertices x , $x+p$, $x+y+p$, $x+y$ (Appendix, fig.2).

Since, $xP^*_r y = (x \circ y) \circ P \supset x \circ P \supset x \circ 0 = x$, we get

$$xP^*_r y \ni x \ , \quad \text{for all} \quad x, y \in \mathbb{R}^2.$$

Proposition 1.

The hyperstructure $\langle \mathbb{R}^2, P^*_r \rangle$ *constructed above is a* $P\text{-}H_v$*-group.*

Proof.

For all $x \in \mathbb{R}^2$,

$$xP^*_r \mathbb{R}^2 = (x \circ \mathbb{R}^2) \circ P = \mathbb{R}^2 \circ P = \mathbb{R}^2 = \mathbb{R}^2 P^*_r x \ .$$

For all $x, y, z \in \mathbb{R}^2$,

$$(xP^*_r y)P^*_r z \supset xP^*_r z \ni x \quad \text{and} \quad xP^*_r (yP^*_r z) \supset xP^*_r y \ni x \ ,$$

so we get

$$[(xP^*_r y)P^*_r z] \cap [xP^*_r (yP^*_r z)] \neq \varnothing \ .$$

3. UNITS AND INVERSES

Let $\blacksquare, \square \in \{r, \ell\}$ where r , ℓ stand for the words "right" and "left" respectively. Let E^r, E^ℓ be the sets of the right and left unit elements respectively, with respect to hyperoperation P^*_r and E be the set of the unit elements. $I^{\blacksquare}_{\square}(x,y)$ is the set of the $\blacksquare$-inverses of x associated with the $\square$-unit y .

Proposition 2.

1) $E^r = \mathbb{R}^2$.

2) $E^\ell = [o, -p] = \{ -\mu p \ / \ \mu \in [0,1] \} = E$.

Proof.

1) Obviously, since $xP^*_{\wedge}y \ni x$, for all $x,y \in \mathbb{R}^2$.

2) Let $e \in E^{\ell}$, then $x \in eP^*_{\wedge}x$ for all $x \in \mathbb{R}^2$, which implies that $x \in \{ e+\lambda x+\mu p \ / \ \lambda,\mu \in [0,1] \}$. The last one means that there exist $\lambda_1,\mu_1 \in [0,1]$ such that $x=e+\lambda_1 x+\mu_1 p$ or $e=(1-\lambda_1)x-\mu_1 p$. So, there exist $\kappa_1,\mu_1 \in [0,1]$: $e=\kappa_1 x-\mu_1 p$.

Since this last equation holds for all $x \in \mathbb{R}^2$, we get that

$$\kappa_1 = 0 \quad \text{and} \quad e = -\mu_1 p \ ,$$

which means that, $e \in \{ -\mu p \ / \ \mu \in [0,1] \} = [0,-p]$.

Due to $E^{\wedge} = \mathbb{R}^2$, it follows that $E^{\ell} = E$.

Remarks.

1) Since $xP^*_{\wedge}y \ni x$, for all $x,y \in \mathbb{R}^2$, each element of $\mathbb{R}^2$ is a left zero element.

2) $E^{\wedge} = E^{\wedge}_{\circ}$, where $E^{\wedge}_{\circ}$ is the set of the right unit elements with respect to the hyperoperation $(\circ)$.

3) Since $E^{\ell}_{\circ}=\{0\}$ and $0 \in E^{\ell} \Rightarrow E^{\ell}_{\circ} \subset E^{\ell}$.

See [2] for the proofs.

The $\blacksquare$-inverse sets of x associated with the $\square$-unit y are given in :

Proposition 3.

1) $I^{\wedge}_{\wedge}(x,y) = \{ \frac{1}{\lambda}(y-x-\mu p) \ / \ \lambda \in (0,1], \ \mu \in [0,1] \}$.

2) $I^{\wedge}_{\ell}(x,y) = \{ \frac{1}{\lambda}(-x-\nu p) \ / \ \lambda \in (0,1], \ \nu \in [0,2] \}$.

3) $I^{\ell}_{\wedge}(x,y) = \{ y-\nu x-\mu p \ / \ \nu,\mu \in [0,1] \}$.

4) $I^{\ell}_{\ell}(x,y) = \{ -\nu x-\mu p \ / \ \nu \in [0,1], \ \mu \in [0,2] \}$.

Proof.

1) Let $x' \in I^{\wedge}_{\wedge}(x,y)$; then $y \in xP^*_{\wedge}x' \Rightarrow y \in [x,x+x'] \circ [0,p] \Rightarrow$ there exist $x'' \in [x,x+x']$ and $p' \in [0,p]$ such that

$$y \in x'' \circ p' = [x'', x''+p'] \ .$$

Since $p' \in [0,p]$, we get $y \in [x'', x''+p] \Rightarrow y-x'' \in [0,p]$.

From $x'' \in [x, x+x'] \Rightarrow y-x'' \in [y-x, y-x-x']$ which means that

$$y-x'' \in \{ \nu p \ / \ \nu \in [0,1] \} \cap \{ y-x-\kappa x' / \ \kappa \in [0,1] \}$$

so, there exist $\nu_1, \kappa_1 \in [0,1]$ such that

$$y-x'' = \nu_1 p \quad \text{and} \quad y-x'' = y-x-\kappa_1 x' .$$

From which, for $\kappa_1 \neq 0$, we get $x' = \frac{1}{\kappa_1}(y-x-\nu_1 p)$. So,

$$x' \in \{ \frac{1}{\lambda}(y-x-\mu p) \ / \ \lambda \in (0,1], \ \mu \in [0,1] \} \ .$$

2) Since $E^{\ell} \subset E^{r} = \mathbb{R}^2$, we have

$x' \in I_{\ell}^{r}(x,y) \Rightarrow x' \in I_{r}^{r}(x,y)$ and $y \in E^{\ell} \Rightarrow$

$$x' \in \{ \frac{1}{\lambda}(y-x-\mu p) \ / \ \lambda \in (0,1], \ \mu \in [0,1] \} \text{ and } y \in \{-\kappa p \ / \ \kappa \in [0,1]\}.$$

So, there exist $\lambda_1 \in (0,1]$ and $\mu_1, \kappa_1 \in [0,1]$ such that

$$x' = \frac{1}{\lambda_1}(y-x-\mu_1 p) \text{ and } y=-\kappa_1 p \Rightarrow x' = \frac{1}{\lambda_1}[-x-(\kappa_1+\mu_1)p] \ ,$$

which means that

$$x' \in \{ \frac{1}{\lambda}(-x-\nu p) \ / \ \lambda \in (0,1], \ \nu \in [0,2] \} \ .$$

3) Analogous to 1).

4) Analogous to 2).

For the geometrical interpretation of the four sets of the above proposition, see Appendix, figs. 4, 5, 6, 7.

Remark.

From the 4^{th} case of the above proposition, we notice that :

$$P' = [0,-2p] = \bigcap_{x \in \mathbb{R}^2} I_{\ell}^{\ell}(x,y) \ ,$$

which means that, for some unit element $y \in E^{\ell}$, the set P' is the common set of the left inverse elements for all $x \in \mathbb{R}^2$.

Proposition 4.

Let $\blacksquare, \square \in \{\wedge, \ell\}$ *then there exist* $x' \in I^{\blacksquare}_{\square}(x,y)$ *such that* $x \in I^{\blacksquare}_{\square}(x',y)$ *for all* $x \in \mathbb{R}^2$.

Proof.

We will prove only the case :

For all $x \in \mathbb{R}^2$, there exists $x' \in I^{\wedge}_{\wedge}(x,y)$ such that $x \in I^{\wedge}_{\wedge}(x',y)$, since the proof for the remaining cases is analogous.

$$x'' \in I^{\wedge}_{\wedge}(x',y) \text{ , where } x' \in I^{\wedge}_{\wedge}(x,y) \Rightarrow$$

$$x'' \in \{ \frac{1}{\lambda}(y-x'-\mu p) \;/\; \lambda \in (0,1],\ \mu \in [0,1] \} \text{ and}$$

$$x' \in \{ \frac{1}{\kappa}(y-x-\nu p) \;/\; \kappa \in (0,1],\ \nu \in [0,1] \}.$$

So, there exist $\lambda_1, \kappa_1 \in (0,1]$ and $\mu_1, \nu_1 \in [0,1]$:

$$x'' = \frac{1}{\lambda_1}(y-x'-\mu_1 p) \text{ and } x' = \frac{1}{\kappa_1}(y-x-\nu_1 p) \Rightarrow$$

$$x'' = \frac{1}{\lambda_1}\left[(1-\frac{1}{\kappa_1})y + \frac{1}{\kappa_1}x + (\frac{\nu_1}{\kappa_1} - \mu_1)p\right] .$$

Finally, there exist $\lambda_1 \in (0,1]$, $\xi_1 \le 0$, $\tau_1 \ge 1$, $\sigma_1 \ge -1$:

$$x'' = \frac{1}{\lambda_1}(\xi_1 y + \tau_1 x + \sigma_1 p) .$$

It means that

$$x'' \in \{ \frac{1}{\lambda}(\xi y + \tau x + \sigma p) \;/\; \lambda \in (0,1],\ \xi \le 0,\ \tau \ge 1,\ \sigma \ge -1 \}.$$

So, $I^{\wedge}_{\wedge}(x',y) = \{\frac{1}{\lambda}(\xi y+\tau x+\sigma p) / \lambda \in (0,1], \xi \le 0, \tau \ge 1, \sigma \ge -1\}$, $x' \in I^{\wedge}_{\wedge}(x,y)$.

Setting $\lambda=\tau=1$, $\xi=\sigma=0$, we get that

$$x \in I^{\wedge}_{\wedge}(x',y) \text{ where } x' \in I^{\wedge}_{\wedge}(x,y).$$

Remarks.

$$I^{\wedge}_{\ell}(x',y) = \{\frac{1}{\lambda}(\mu x+\nu p) \;/\; \lambda \in (0,1],\ \mu \ge 1,\ \nu \ge -2\} \text{ , } x' \in I^{\wedge}_{\ell}(x,y).$$

$$I^{\ell}_{\wedge}(x',y) = \{\lambda y+\mu x+\nu p) \;/\; \lambda, \mu \in [0,1],\ \nu \in [-1,1]\} \text{ , } x' \in I^{\ell}_{\wedge}(x,y).$$

$$I^{\ell}_{\ell}(x',y) = \{\lambda x+\mu p) \;/\; \lambda \in [0,1],\ \mu \in [-2,2]\} \text{ , } x' \in I^{\ell}_{\ell}(x,y).$$

A **Join Space** (see [3]) is a commutative hypergroup $\langle J, \cdot \rangle$ with a hyperoperation $/ : J\times J \longrightarrow \mathcal{P}(J)$ defined by

$$a/b=\{x\in J: a\in x\cdot b\} \quad \text{for all} \quad a,b \in J ,$$

which satisfies the "transposition law" :

$$a,b,c,d \in J \quad \text{and} \quad a/b \cap c/d \neq \emptyset \quad \text{implies} \quad a\cdot d \cap b\cdot c \neq \emptyset.$$

Proposition 5.

For all $a,b,c,d \in \mathbb{R}^2$, *the following implication is valid :*

$$(a / b) \cap (c / d) \neq \emptyset \Rightarrow (aP^*_{\Lambda}d) \cap (bP^*_{\Lambda}c) \neq \emptyset .$$

Proof.

Let $(a / b) \cap (c / d) \neq \emptyset$. Then there exists $x\in\mathbb{R}^2$ such that

$$a \in xP^*_{\Lambda}b \quad \text{and} \quad c \in xP^*_{\Lambda}d .$$

$a \in xP^*_{\Lambda}b$ implies that $a \in \{ x+\lambda b+\nu p \ / \ \lambda,\nu\in[0,1] \}$.

Similarly, $c \in \{ x+\kappa d+\xi p \ / \ \kappa,\xi\in[0,1] \}$.

So, there exist $\lambda',\nu',\kappa',\xi'\in [0,1]$ such that

$$a = x+\lambda' b+\nu' p \quad \text{and} \quad c = x+\kappa' d+\xi' p .$$

Therefore $d = \frac{1}{\kappa'}(-x+c-\xi' p)$, $\kappa'\neq 0$.

Now

$$aP^*_{\Lambda}d = \{ a+\sigma d+\theta p \ / \ \sigma,\theta \in [0,1] \} =$$

$$= \{ x+\lambda' b+\nu' p + \frac{\sigma}{\kappa'}(-x+c-\xi' p)+\theta p \ / \ \sigma,\theta \in [0,1] \} =$$

$$= \{(1- \frac{\sigma}{\kappa'})x + \lambda' b + \frac{\sigma}{\kappa'}c + (\nu' - \frac{\sigma\xi'}{\kappa'} + \theta)p \ / \ \sigma,\theta \in [0,1]\} .$$

For $\sigma=\kappa'\lambda'$, $x=b$ and $\theta=\lambda'\xi'$ we get $aP^*_{\Lambda}d \ni b+\lambda' c+\nu' p$.

Since, $bP^*_{\Lambda}c = \{ b+\delta c+\zeta p \ / \ \delta,\zeta \in [0,1] \}$, for $\delta=\lambda'$ and $\zeta=\nu'$, we get that $bP^*_{\Lambda}c \ni b+\lambda' c+\nu' p$. So,

$$(aP^*_{\Lambda}d) \cap (bP^*_{\Lambda}c) \neq \emptyset .$$

Remark.

For $a,c \in E^{\Lambda}$ we have

$$a / b = I^{\ell}_{\Lambda}(b,a) \quad \text{and} \quad c / d = I^{\ell}_{\Lambda}(d,c).$$

Fig.8 in Appendix gives the geometric interpretation of the above proposition.

APPENDIX

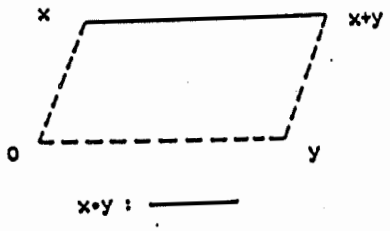


Fig.1

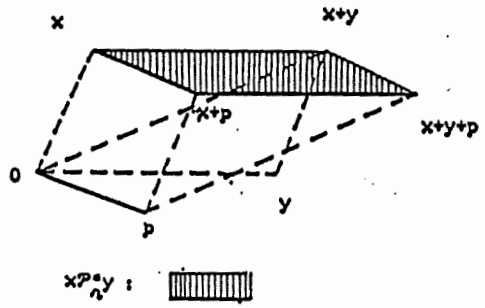


Fig.2

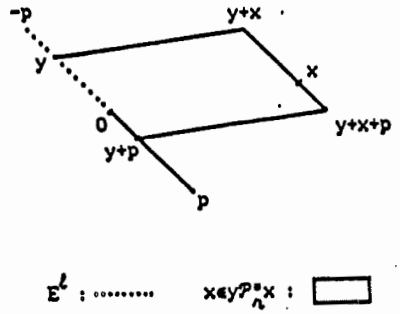


Fig.3

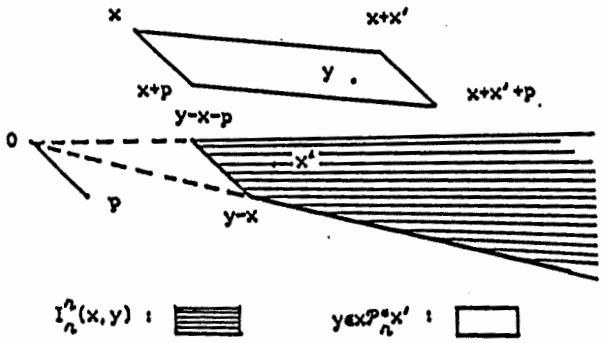
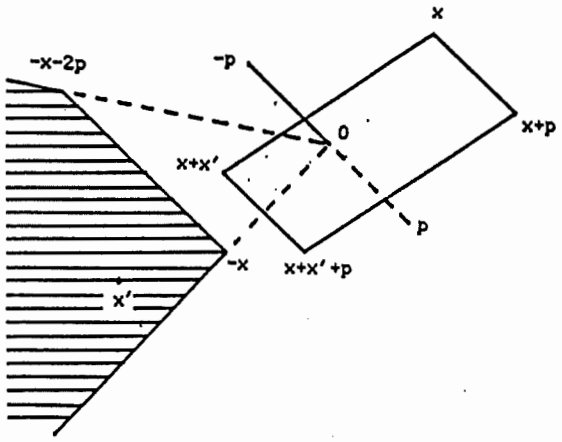
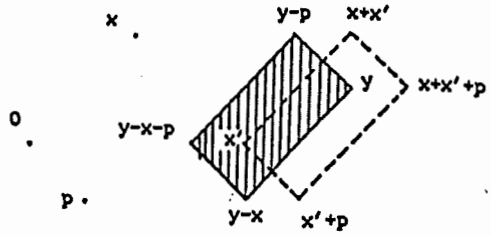


Fig.4



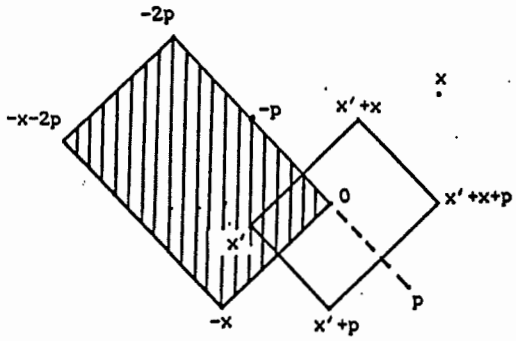
$I_\ell^{\mathcal{P}_\Lambda}(x,y)$: $y \in x \mathcal{P}_\Lambda^* x'$:

Fig. 5



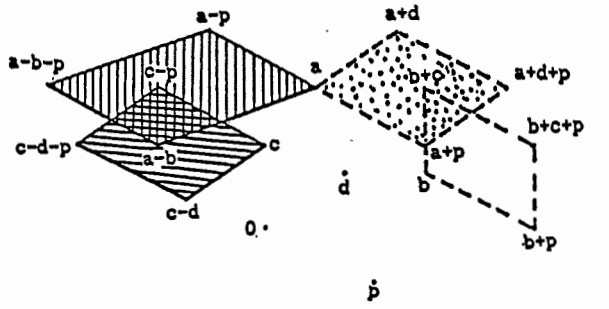
$I_\Lambda^\ell(x,y)$: $y \in x' \mathcal{P}_\Lambda^* x$:

Fig. 6



$I_\ell^{\mathcal{P}_\Lambda}(x,y)$: $y \in x' \mathcal{P}_\Lambda^* x$:

Fig. 7



a / b : c / d : $a \mathcal{P}_\Lambda^* d$: $b \mathcal{P}_\Lambda^* c$:

Fig. 8

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CONSTRUCTION OF H_v-STRUCTURES WITH DESIRED FUNDAMENTAL STRUCTURES

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Abstract

In this paper we deal with the class of the **H_v-structures** where the classical axioms are replaced by weak ones. The **weak axioms** are the ones where the non-empty intersection replaces equality. In every H_v-structure the fundamental relation leads to the corresponding fundamental structure.

We focus our attention on constructions which have a given structure as the fundamental one. Moreover, we study a characteristic class of H_b-structures.

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1. INTRODUCTION

In this paper we deal with the largest class of hyperstructures, the **H_v-structures**, introduced in 1990 [2]. They satisfy the **weak axioms** where the non-empty intersection relation replaces the equality relation. For example, the **weak associativity**, (abbr. **WASS**), in a set H equipped with a hyperoperation $(\cdot)$, is

$$(xy)z \cap x(yz) \neq \emptyset, \quad \forall x,y,z \in H$$

and the **weak commutativity**, (abbr. **COW**), is

$$xy \cap yx \neq \emptyset, \quad \forall x,y \in H.$$

DEFINITIONS 1.1

A pair $(H,\cdot)$ is called a **H_v-semigroup** if $(\cdot)$ is WASS, a **H_v-group** if $(\cdot)$ is WASS and reproductive, a **H_v-abelian group** if $(\cdot)$ is WASS, COW and reproductive, and it is called an **abelian H_v-group** if the commutativity is valid instead of COW.

A triple $(R,+,\cdot)$ is called a **H_v-ring** if $(R,+)$ is H_v-group, $(R,\cdot)$ is H_v-semigroup and $(\cdot)$ is weak distributive to $(+)$. Matrices with entries that are elements of a H_v-ring are called **hypermatrices**. Matrix multiplication is defined in the usual manner. A H_v-group $(M,\oplus)$ is called a **H_v-module** over the H_v-ring $(R,+,\cdot)$ if an external hyperoperation $(\circ)$ is defined which satisfies the following distributive axioms for all $r,s \in R$ and $x,y \in M$:

$$r\circ(x\oplus y) \cap (r\circ x \oplus r\circ y) \neq \emptyset \quad \text{and} \quad (r+s)\circ x \cap (r\circ x \oplus s\circ x) \neq \emptyset$$

and

$$(r\cdot s)\circ x \cap r\circ(s\circ x) \neq \emptyset.$$

Let $(G,\cdot)$, $(G,\odot)$ be two H_v-group, defined on the same set G. We call $(\cdot)$ **smaller** than $(\odot)$, and we write $\cdot \leq \odot$, iff there is a $f \in Aut(G,\odot)$ such that

$$xy \subseteq f(x\odot y) \quad \text{for all } x,y \text{ in } G.$$

We also say that $(\odot)$ is **greater** than $(\cdot)$. Furthermore, we say

that $(G,\odot)$ contains the $(G,\cdot)$. We can also have analogous definitions for the other H_v-structures we can consider.

If a H_v-group $(G,\odot)$ contains a group $(G,\cdot)$ then $(G,\odot)$ is called a **H_b-group** and $(G,\cdot)$ **a basic group**. Definitions for the remaining H_b-structures are given in a similar way.

In section 2 a fundamental relation is defined in every H_v-structure and leads, by a quotient, to a corresponding classical structure.

In this paper we present new constructions of structures which have a given funtamental structure. Moreover, the class of very thin hyperstructures with basic structure is investigated in section 4.

2. THE FUNDAMENTAL STRUCTURES

DEFINITIONS 2.1

Let $(H,\cdot)$ be H_v-group, then the **fundamental relation** β^* is the smallest equivalence relation such that the quotient H/β^* is a group. Let $(R,+,\cdot)$ be H_v-ring, then the **fundamental relation** γ^* is the smallest equivalence relation such that the quotient R/γ^* is a ring. If the fundamental ring R/γ^* is a field then $(R,+,\cdot)$ is called **H_v-field** and its elements are called **hypernumbers**. Let $(M,\circledast)$ be H_v-module over the H_v-ring $(R,+,\cdot)$, then the **fundamental relation** ε^* is the smallest equivalence relation such that the quotient M/ε^* is a module over the ring R/γ^*. If a H_v-module is defined over the H_v-field then $(M,\circledast)$ is called a **H_v-vector space**. The above structures H/β^*, R/γ^* and M/ε^* are called the **fundamental structures** of the hyperstructures H, R and M respectively.

An element of a H_v-structure is called **single** if the class of its fundamental relation contains only this element. The set of the single elements of H is denoted by S_H.

The fundamental relations are the transitive closure of some special relations. We give this relation for H_v-groups. Similar theorems can be obtained for the remaining cases.

Let $(H,\cdot)$ be H_v-group and let $\mathcal{U}$ denote the set of all finite products of elements of H. If the relation β in H is defined as

$$x\beta y \quad \text{iff} \quad \{x,y\} \subset u \quad \text{where} \quad u\in\mathcal{U},$$

then the fundamental relation β^* is the transitive closure of the relation β.

In a similar way, as the definition of H_v-vector space, one can define other stricter hyperstructures.

From the definition of the H_v-field, it is clear that some general definitions of H_v-structures can be given only through the fundamental relations. It is known that a powerful structure in Physics is the Lie algebra. Therefore, in this section we give the general definition of the H_v-Lie algebra.

DEFINITION 2.2

Let $(L,+)$ be a H_v-vector space over the H_v-field $(F,+,\cdot)$. Take the canonical map $\varphi: F \longrightarrow F/\gamma^*$ and denote $\omega_F = \{x\in F: \varphi(x)=0\}$, where 0 is the zero element of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi': L \longrightarrow L/\varepsilon^*$ and denote again, by the same symbol 0 the zero element of L/ε^*. Consider any **bracket** or **commutator** hyperoperation:

$$[,] : L\times L \longrightarrow \mathcal{P}(L) : (x,y) \longmapsto [x,y]$$

Then L is a **H_v-Lie algebra** over F if the following axioms are satisfied:

(L1) The bracket hyperoperation is bilinear, i.e.

$$[\lambda_1 x_1+\lambda_2 x_2, y] \cap (\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \varnothing$$

$$[x, \mu_1 y+\mu_2 y] \cap (\mu_1[x,y_1]+\mu_2[x,y_2]) \neq \varnothing$$

for all $x, x_1, x_2, y, y_1, y_2 \in L$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \emptyset$ for all $x \in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]]) \cap \omega_L \neq \emptyset$ for all $x,y,z \in L$.

The above definition of the H_v-Lie algebra is the most general. Therefore, in research, one can take special cases of this general definition. The special cases can be either taking some hyperoperations as operations or, taking some special elements. In the last case one can assume that the zero classes of L and F are singletons. Thus, denoted by the same symbol 0, these two elements of F and L are singles. Then the relations (L2) and (L3) become

$$[x,x] \ni 0 \quad \text{and} \quad [x,[y,z]]+[y,[z,x]]+[z,[x,y]] \ni 0$$

Some nice properties can be obtained for these H_v-Lie algebras.

From the fact that a new hyperoperation $[,]$ is defined in L, taking into account the new hyperoperation in addition to the old ones, there exists a new fundamental relation $\underline{\varepsilon}^*$. Obviously $\varepsilon^*(x) \subset \underline{\varepsilon}^*(x)$ for all $x \in L$. Denote $\underline{S}_L$ the set of single elements with respect to $\underline{\varepsilon}^*$. Finally, remark that if $[x,y] \subset \varepsilon^*(z)$, then $\underline{\varepsilon}^*(z) = \varepsilon^*(z)$ for all $z \in L$.

If $0 \in \underline{S}_L$ then, the above relations (L2) and (L3) become

$$[x,x] = 0 \quad \text{and} \quad [x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0.$$

This is immediate from the facts that 0 belongs to the left side sets and $0 \in \underline{S}_L$.

DEFINITION 2.3

The class of H_v-Lie algebras with $0 \in \underline{S}_L$ will be called **H_v^o-Lie algebras** and it is the most interesting class.

If $\varphi: H \longrightarrow H/\beta^*$ is the natural map of a H_v-group $(H,\cdot)$, then denote by ω_H the **core** of H, i.e. the kernel of φ.

Now, recall some useful theorems [3], [4]:

THEOREM 2.4

Let $(H,\cdot)$ be a H_v-group and $s\in S_H$. Then,

$$\omega_H=\{u\in H:\ us=s\}=\{u\in H:\ su=s\}.$$

Let $a\in H$ and take a $v\in H$ such that $s\in av$. Then

$$\beta^*(a)=\{h\in H:\ hv=s\}.$$

THEOREM 2.5

Let $(H,\cdot)$ be a H_v-group and $s\in S_H$, then

$$sx = \beta^*(sx) \quad \text{and} \quad xs = \beta^*(xs) \quad \text{for all } x\in H.$$

THEOREM 2.6

Proper H_b-groups have no single elements.

DEFINITION 2.7

Let $(H,\cdot)$ be a H_v-group. An element $x\in H$ is called **left complete** if $xy=\beta^*(xy)$ for all $y\in H$. Similarly, the **right complete** and the **complete** elements are defined. If all elements of H are complete elements, then H is called complete [1].

Note that all single elements in a H_v-group are complete but the opposite is not always true. Finally, we do not know if and when in other H_v-structures the single elements are complete.

3. NEW CONSTRUCTIONS (DH_v-STRUCTURES)

DEFINITION 3.1

Let $(\mathbf{G},\cdot)$ be a semigroup. Consider a pairwise disjoint family of sets $\{\mathbf{A}_g \mid g\in\mathbf{G}\}$ with indexes from $\mathbf{G}$ such that each $\mathbf{A}_g$ contains the element which is its index. We set $\mathbf{A} = \bigcup_{g\in G} \mathbf{A}_g$. A non-degenerate hyperoperation $(\odot)$ defined on the set $\mathbf{A}$ is called a **DH_v-operation** if it satisfies:

$$x\odot y \subseteq \mathbf{A}_{g_1g_2} \qquad \text{for all } x\in\mathbf{A}_{g_1},\ y\in\mathbf{A}_{g_2},\ g_1,g_2\in\mathbf{G}$$

Every hyperstructure $(\mathbf{A},\odot)$ where a DH_v-operation is defined is called a **DH_v-structure**.

EXAMPLES 3.2

(a) An example of DH_v-group is the following generalization of DeSalvo's (H,G)-group [1]:

Let $<H,*>$ be a H_v-group, $(G,\cdot)$ be a group with identity 1, $\{A_i\}_{i\in G}$ be a family of non-empty sets indexed of G with $A_1=H$.

Set $\mathbf{K}=\bigcup_{i\in G} A_i$ and consider the hyperoperation $(\odot)$ defined in **K** as follows

$$x\odot y = x*y \quad \text{for all} \quad x,y\in H$$

$$x\odot y = A_{i\cdot j} \quad \text{for all} \quad (x,y)\in A_i\times A_j \neq H\times H.$$

Then $<H,\odot>$ becomes a H_v-group, which is called a **(H,G)-H_v-group**. It is easy to see that $\mathbf{K}/\beta^* \cong G$. Moreover, generalizations of this construction on H_v-rings etc, can be defined similarly.

(b) Let $(S_i,\cdot)$, $i\in I$, $|I|>1$ be a pairwise disjoint family of H_v-semigroups. Then the set $\mathbf{S}=\bigcup_{i\in I} S_i$ becomes a H_v-group with respect to the following hyperoperation

$$x_i\odot y_i = x_i y_i \quad \text{for all } (x_i,y_i)\in S_i^2$$

$$x_i\odot x_j = S_i\cup S_j \quad \text{for all } (x_i,x_j)\in S_i\times S_j,\ i\neq j.$$

This construction is called a **S-construction** and we have $|\mathbf{S}/\beta^*|=1$. This is not a DH_v-structure since it is not based on a group and for different S_i's the hyperproduct is greater than the ones defined in a DH_v-structure.

PROPOSITION 3.3

Suppose that the H_v-groupoid $(\mathbf{A},\odot)$ is WASS. Then for every $a\in\mathbf{A}_g$ we have $\beta^*(a)\subset\mathbf{A}_g$. For finite **G** we have $|\mathbf{A}/\beta^*|\geq|\mathbf{G}|$.

Proof

Suppose there exists a finite product $\underset{i\in I}{\odot} x_i$ of elements $x_i \in \mathbf{A}_{g_i}$ for all $i\in I$, such that

$$\{a,b\} \subset \underset{i\in I}{\odot} x_i \ .$$

Then, by the definition, it is obtained that $\{a,b\} \subset \mathbf{A}_{\underset{i\in I}{\odot} g_i}$.

Applying the transitive closure it is obtained that the fundamental class $\beta^*(a)$ of any element $a\in\mathbf{A}_g$ is a subset of the only one set $\mathbf{A}_g$.

CONSTRUCTION 3.4

Let $(\mathbf{G},\cdot)$ be a group. Consider the pairwise disjoint family of sets $\{\mathbf{A}_g \mid g\in\mathbf{G}\}$ with indexes from $\mathbf{G}$ and assume each $\mathbf{A}_g$ contains the element g which is its index. Then the set $\mathbf{A} = \bigcup_{g\in G} \mathbf{A}_g$, becomes a H_v-group with respect to **every** DH_v-operation $(\odot)$ satisfying the following:

$$g_1g_2 \subseteq x\odot y \subseteq \mathbf{A}_{g_1g_2} \quad \text{for all } x\in\mathbf{A}_{g_1},\ y\in\mathbf{A}_{g_2},\ g_1,g_2\in\mathbf{G} \qquad \text{(i)}$$

and

$$x\odot\mathbf{A}_{g'} = \mathbf{A}_{gg'} \quad \text{for all } x\in\mathbf{A}_g,\ g,g'\in\mathbf{G}. \qquad \text{(ii)}$$

The H_v-group $(\mathbf{A},\odot)$ has $\mathbf{A}/\beta^* \cong \mathbf{G}$.

Every H_v-group $(\mathbf{A},\odot)$ defined as above is called a **DH_v-group**.

Proof

Every $\odot$ is WASS since for all $x\in\mathbf{A}_{g_1}$, $y\in\mathbf{A}_{g_2}$, $z\in\mathbf{A}_{g_3}$ we have

$$(x\odot y)\odot z \supset g_1g_2\odot z \supset g_1g_2g_3 \quad \text{and} \quad x\odot(y\odot z) \supset x\odot(g_2g_3) \supset g_1g_2g_3.$$

Therefore, $(x\odot y)\odot z \cap x\odot(y\odot z) \neq \emptyset$.

Moreover, every $\odot$ is reproductive since for all $x\in\mathbf{G}$ such that $x\in\mathbf{A}_g$ we have

$$x\odot\mathbf{A} = x\odot\Big(\bigcup_{g'\in G}\mathbf{A}_{g'}\Big) = \bigcup_{g'\in G}\mathbf{A}_{gg'} = \bigcup_{g'\in G}\mathbf{A}_{g'} = \mathbf{A}\ .$$

From Proposition 3.3 we know that $\beta^*(a) \subset \mathbf{A}_g$. Moreover, take any $a\in\mathbf{A}_g$ and two elements $g_1, g_2\in\mathbf{G}$ such that $g_1\cdot g_2=g$. Then, from the above two hypotheses, there exists an element $x\in g_1$ such that $x\odot g_2 \supset \{g_1\cdot g_2,\ a\}$. So $a\in\beta^*(g)$ which means that $\beta^*(a) \supset \mathbf{A}_g$. Therefore, $\beta^*(a) = \mathbf{A}_g$ for all $a\in\mathbf{A}_g$. In other words we have $\mathbf{A}/\beta^* \cong \mathbf{G}$.

REMARKS 3.5

(a) A special case for the relation (ii) in Construction 3.4 is the following:

In every set $A_g, g\in G$, suppose there exists at least one complete element. Notice that some of them can be single elements.

(b) The Construction 3.4 can be applied in H_v-rings, H_v-vector spaces etc. The relation (i) must be valid for every hyper-operation and it guarantees for the WASS axioms. For the remaining axioms one can put relations like the (ii).

(c) Let us denote by e the unit of G. Then for every $x\in A_e$ we must have $x\odot A_e=A_e$. Therefore, the set A_e must be a sub-H_v-group. This remark is applied to (H,G)-H_v-groups, in fact one supposes that H is sub-H_v-group in this construction. Finally, the same remark is also applicable in every idempotent element of G. Thus for every $x\in G$ such that $x\cdot x=x$ we obtain that A_x is a sub-H_v-group.

Since the single elements are important especially if they are characteristic as unit or zero elements. We give the following construction of a DH_v-group. Similar ones for other DH_v-structures can be defined.

CONSTRUCTION 3.6

Let $(G,\cdot)$ be a group and e its unit. Consider a pairwise family of sets $\{\mathbf{A}_g \mid g\in G\}$ with indices from G, such that $g\in\mathbf{A}_g$, $\forall\, g\in G$ and $\mathbf{A}_e=\{e\}$. Then, the set $\mathbf{A} = \bigcup_{g\in G} \mathbf{A}_g$, is DH_v-group with respect to every DH_v-operation $(\odot)$ which satisfy the relations (i), (ii) and e is a single element.

The proof is obvious.

This construction enjoys the following properties:

PROPERTIES 3.7

(a) For every $x\in\mathbf{A}_g$ the set of inverse elements of x with respect to $(\odot)$, is $\mathbf{A}_{g^{-1}}$ and for all $y\in\mathbf{A}_{g^{-1}}$ we have $x\odot y=y\odot x=e$. Otherwise, $\mathbf{A}_g\odot\mathbf{A}_{g^{-1}} = \{e\}$.

(b) Suppose that $(G,\cdot)$ is cyclic, $g\in G$ is a generator and $|G|=n$. Then every element $x\in\mathbf{A}_g$ is a generator with period r such that $n \le r \le 2n$.

Proof

(a) Obvious.

(b) We have $x^n=e$, so $x^{n+1} = \mathbf{A}_g$ and then

$$x^{n+2} = \mathbf{A}_{g^2}\,, \quad x^{n+3} = \mathbf{A}_{g^3}\,, \quad \ldots$$

Therefore the period r satisfy the relation $n\le r\le 2n$.

4. VERY THIN H_b-STRUCTURES

Every structure can be considered as a minimal H_b-structure, however it is not a proper hyperstructure. Very thin H_b-structures are those where the one of the products $x\cdot y$ of the b-structure is enlarged. The minimal proper very thin H_b-structure are the ones in which one special product $a\cdot b$ is enlarged by only one element d. Thus the only hyperproduct is $a\odot b = \{a\cdot b,\ d\}$ and the rest are usual products. This is the case we will study in this section. The reason we choose this case is that this is similar to the chain reaction in Physics.

First we point out the general property of H_b-structures which is obvious:

PROPERTY 4.1

The set [S] generating by any subset S of a H_b-structure H is always a sub-H_b-structure.

THEOREM 4.2

All proper H_b-fields have only one fundamental class.

Proof

Let $(F,\oplus,\odot)$ be H_b-field and $(F,+,\cdot)$ be a basic field. Take two elements $a,b\in F$, $a\neq b$ such that there exists a hyperproduct

$$c\odot d \supset \{a,b\}.$$

If $-b$ is the opposite element of b in $(F,+,\cdot)$, then we have

$$c\odot d+(-b) \supset \{ a-b,\ 0 \}.$$

This means that $a-b\in\beta^*(0)$.

Moreover, from the reproductivity in $F-\{0\}$ we have

$$(c\odot d+(-b))\cdot x \supset \{ (a-b)\cdot x,\ 0 \} \quad \text{for all } x\in F-[0].$$

Thus $(F,\oplus,\odot)$ has only one fundamental class.

Now we restrict ourselves to H_b-structures $(H,\cdot)$ equipped with only one hyperoperation. Moreover, we remark that in this case, from Theorem 2.5, in the H_v-groups we deal with there are no single elements.

We fix the two elements $a,b\in H$ such that $a\cdot b=c$, as well as the element $d\in H$ such that $a\odot b = \{c,\ d\}$.

THEOREM 4.3

Let $(H,\odot)$ be a very thin H_b-group, with the basic group $(H,\cdot)$ and with the only one hyperproduct, $a\odot b=\{c,d\}$, where the elements a,b,d are fixed and $a\cdot b=c$. Then the core $\omega_H=\beta^*(e)$ is equal to the subgroup generated by the set

$$\mathbf{D} = \{c^{-1}d,\ dc^{-1},\ d^{-1}c,\ cd^{-1}\}.$$

Moreover, we have

$$\beta^*(x) = x\beta^*(e) \quad \text{for every } x\in H.$$

Proof

First we observe, from the fact that $(H,\odot)$ is a H_b-group, that

$$\text{if } A \subset \beta^*(x) \text{ for some } x \in H, \text{ then } yA \subset \beta^*(yx), \quad \forall y \in H.$$

Therefore, in the case of H_b-groups, in order to compute the fundamental classes, we have to compute the class $\beta^*(e)=\omega_H$. Moreover, in our case, in order to find the β^*-related elements we have to use the hyperproduct $a \odot b$.

Let us multiply the hyperproduct $a \odot b$, consequently by c^{-1} and d^{-1} on the left and on the right. Then, using the trasitivity, it is obtained that $\mathbf{D} \subset \beta^*(e)$.

Now, in order to obtain more new elements β^*-related to e we have to multiply by $a \odot b$. Therefore, it is obvious, that the core ω_H is exactly the set generated by the set **D**.

Thus, we have

$$\omega_H = \beta^*(e) = [c^{-1}d,\ dc^{-1},\ d^{-1}c,\ cd^{-1}].$$

Finally, the fundamental class of any element $x \in H$ is, from the first remark, $x\beta^*(e)$.

This theorem proves that this "chain reaction" is inside the fundamental classes with respect to the set $\beta^*(e)$.

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